# 7350: TOPICS IN FINITE-DIMENSIONAL ALGEBRAS

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Abstract. These are the notes from a course taught at Cornell in Spring 2009. They are a record of what was covered in the lectures. Many results were copied out of books and other sources, as noted in the text. The course had three parts: (1) An introduction to finite-dimensional algebras (lectures 1-12) (2) Gabriel's Theorem (lectures 13-16) (3) Auslander-Reiten theory (lectures 17-27).

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## 1. Lecture 1

1.1. **Introduction.** This course is about finite-dimensional algebras. An algebra is a vector space in which you can multiply the vectors (a formal definition will be given below). Alternatively, an algebra is a ring which happens to be a vector space.

We work over a field $k$, which we usually take to be algebraically closed of characteristic zero, and usually just assume that $k = \mathbb{C}$.

List of references: (more to be added)

[ARS97], [ASS06] are good books on finite-dimensional algebras; [Rot09] is a good reference for homological algebra; [Bea99] is good for basic ring theory (eg. Jacobson radical); [MR01] is a good reference for ring theory in general; [CB] is an extremely good set of notes on quivers available online, which will be used for some parts of the course. See the same webpage for other sets of notes which are also worth reading.

Why study finite-dimensional algebras? Answer: they are popular and interesting. We are really interested in the module categories of such algebras, not the algebras themselves. It turns out that a lot of interesting abelian categories are actually the category of modules over some finite-dimensional algebra $A$. For example the category $\mathcal{O}(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple complex Lie algebra.

1.2. **Basic definitions.**

**Definition 1.1.** *A $k$–*algebra *is a $k$–vector space $A$ together with two linear maps*

$$m : A \otimes_k A \to A$$

$$\eta : k \to A$$

*satisfying the axioms*

$$m(m \otimes id) = m(id \otimes m) : A \otimes A \otimes A \to A$$

$$m(\eta \otimes id) = id : A = k \otimes_k A \to A$$

$$m(id \otimes \eta) = id : A = A \otimes_k k \to A.$$

In more sensible language, $m$ is a bilinear map from $A \times A$ to $A$, $\eta$ is an element $\eta(1) \in A$, and if we write $m(a \otimes b) = ab$ and $\eta(1) = 1_A$, then the axioms say that

$$a(bc) = (ab)c$$

$$a1_A = a$$

$$1_A a = a$$

for all $a, b, c \in A$.

Here are some examples.

- $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathbf{i}$ is an $\mathbb{R}$–algebra.
- $M_n(k) = \{n \times n$ matrices over $k\}$ is a $k$–algebra.
- $k[x_1, x_2, \ldots, x_n]$.
- $kG$, $G$ a group.

**Definition 1.2** (Morphisms)**.** *A morphism $f : A \to B$ of algebras is a linear map $f : A \to B$ such that $f(a_1a_2) = f(a_1)f(a_2)$ for all $a_1, a_2 \in A$ and $f(1_A) = 1_B$. In terms of our fancier definition, these would be written $fm_A = m_B(f \otimes f)$ and $f\eta_A = \eta_B$.*

**Definition 1.3** (Ideals)**.** *A* left ideal *of $A$ is a subspace $I \subset A$ such that $aI \subset I$ for all $a \in A$ and a* right ideal *is a subspace $I$ such that $Ia \subset I$ for all $a \in A$. A* two-sided ideal *is a subspace which is both a left and a right ideal.*

*If $I$ is a two-sided ideal then the quotient $A/I$ becomes an algebra with the appropriate definition of multiplication and unit element.*

**Definition 1.4** (Product)**.** *If $A$ and $B$ are algebras then $A \times B$ denotes the vector space $A \oplus B$ with the product*

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$$

*and identity $(1_A, 1_B)$. This is a product in the category of algebras (exercise).*

**Definition 1.5** (Tensor product)**.** *If $A$ and $B$ are algebras then $A \otimes B$ denotes the vector space $A \otimes_k B$ with the product*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$$

*and identity $1_A \otimes 1_B$.*

**Definition 1.6** (Opposite). *If $A$ is an algebra then $A^{op}$ is the algebra with underlying vector space $A$ and multiplication $m_{A^{op}}(a_1 \otimes a_2) = m_A(a_2 \otimes a_1)$.*

**Definition 1.7** (Free algebra). *If $V$ is a vector space, define*

$$T(V) = \bigoplus_{i=0}^{\infty} \underbrace{(V \otimes V \otimes \cdots \otimes V)}_{i \text{ copies}}$$

*where the $i = 0$ component is just $k$. The vector space $T(V)$ can be equipped with a multiplication given on the obvious basis by concatenation. This is a well-defined algebra. If $v_1, v_2, \ldots, v_n$ is a basis of $V$ then $T(V)$ has a basis consisting of all words in the $v_i$. It is known as the* tensor algebra *of $V$ or the* free algebra *on $v_1, \ldots, v_n$ and is also denoted $k\langle v_1, v_2 \ldots, v_n\rangle$.*

**Exercise 1.8.**
(1) *Show that $T(V)$ is a well-defined algebra.*
(2) *Show that every algebra can be written $T(V)/I$ for some vector space $V$ and some two-sided ideal $I$ of $T(V)$.*

### 1.3. **Modules.**

**Definition 1.9.** *If $A$ is a $k$–algebra, a* left $A$–module *is a vector space $M$ together with a linear map $\alpha : A \otimes_k M \to M$ such that*

$$\alpha(m_A \otimes id) = \alpha(id \otimes \alpha).$$

*That is, $a(bn) = (ab)n$ for all $a, b \in A$ and all $n \in M$, where $an$ denotes $\alpha(a \otimes n)$.*

*If $M$, $N$ are left $A$–modules, then a* module map *$f : M \to N$ is a linear map which commutes with the $A$–actions, in the sense that $f(an) = af(n)$ for all $a \in A$ and all $n \in M$.*

There is an analogous definition of right $A$–module.

**Definition 1.10.** *We denote by*

$$A - \mathrm{Mod}$$

*the category of all left $A$–modules and by*

$$A - \mathrm{mod}$$

*the category of all* finite-dimensional *left $A$–modules.*

We also use the notation $\mathrm{Mod} - A$ and $\mathrm{mod} - A$ for right $A$–modules.

Notice that $\mathrm{mod} - A$ is equivalent to $A^{op} - \mathrm{mod}$. Also, there is the notion of $A - A$–bimodule, which is the same thing as a left module for $A \otimes A^{op}$. We usually think of this as a vector space with a left and right action of $A$ such that the two actions commute with each other. We leave it to the reader to make a proper definition of bimodule.

The category $A - \mathrm{mod}$ is the main object of study in this course. By convention, when we talk about a "module" without qualification, we mean a finite-dimensional left module.

Surprisingly, there is quite a lot to be said about $A-\text{mod}$ for a general finite-dimensional algebra $A$. This will be explained in the next few lectures.

It is assumed that you have taken a course in homological algebra and are familiar with the notion of submodule, quotient, kernels, cokernels, direct sums, and short exact sequences.

From now on, all algebras are assumed to be finite-dimensional except when stated otherwise.

### 1.4. **Simple modules.**

**Definition 1.11.** *A module $M$ is called* simple *if the only submodules of $M$ are $0$ and $M$.*

For example, $\mathbb{C}^n$ is a simple $M_n(\mathbb{C})$–module because given any nonzero vector, you can find a matrix which takes it to any other nonzero vector.

Every module can be built up out of simple modules in the following sense

**Definition 1.12.** *Let $M$ be a module. A* composition series *of $M$ is a sequence of submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

*such that the modules $M_i/M_{i-1}$ are simple for all $i$.*

Every module has a composition series. To see this, given a finite-dimensional $M$, let $M_1$ be a nonzero submodule of $M$ of smallest possible dimension. Then $M_1$ is simple. Now, if $M_1 \neq M$, let $M_2' \subset M/M_1$ be a nonzero submodule of smallest possible dimension. Let $M_2 \subset M$ be a submodule such that $M_2' = M_2/M_1$. Continue like this, and you will have constructed a composition series of $M$. (As an exercise, make this proof precise).

**Definition 1.13.** *If $M$ is a module and*

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

*is a composition series, then $n$ is called the* length *of the series and the $M_i/M_{i-1}$ are called the* composition factors *of the series.*

In fact, the length and the isomorphism classes of the composition factors are uniquely determined by $M$. This is the content of our first important theorem.

## 2. Lecture 2

### 2.1. **The Jordan-Hölder Theorem.**

**Definition 2.1.** *If $M$ is a module, define the* length *$\ell(M)$ to be the minimum length of a composition series of $M$.*

**Lemma 2.2.** *If $N$ is a proper submodule of $M$, then $\ell(N) < \ell(M)$.*

*Proof.* Let

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

be an arbitrary composition series for $M$. Then

$$0 = N \cap M_0 \subset N \cap M_1 \subset N \cap M_2 \subset \cdots \subset N \cap M_n = N$$

is a series of submodules of $N$. The factors of this series are $N \cap M_i / N \cap M_{i-1}$. The natural map $N \cap M_i / N \cap M_{i-1} \to M_i / M_{i-1}$ is an injection, and therefore, since $M_i/M_{i-1}$ is simple, we have that either $N \cap M_i / N \cap M_{i-1} = M_i/M_{i-1}$ is simple, or else $N \cap M_i / N \cap M_{i-1} = 0$. It suffices to show that there must be at least one $i$ with $N \cap M_i / N \cap M_{i-1} = 0$. To see this, observe that

$$\dim(N) = \sum_i \dim(N \cap M_i / N \cap M_{i-1}).$$

If $N \cap M_i / N \cap M_{i-1} = M_i / M_{i-1}$ for all $i$, then we get $\dim(N) = \dim(M)$ and so $N$ cannot be a proper submodule. □

**Theorem 2.3** (Jordan-Hölder Theorem)**.** *Any two composition series of a module $M$ have the same length and the same composition factors, up to isomorphism.*

*Proof.* This is proved by induction on $\ell(M)$. If $\ell(M) = 1$ then $M$ is simple. The theorem is obvious in this case.

Suppose the result is true for all modules $N$ with $\ell(N) < \ell(M)$. Suppose we have two composition series of $M$.

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

and

$$0 = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_{p-1} \subsetneq K_p = M.$$

If $M_{n-1} = K_{p-1}$ then Lemma 2.2 allows us to apply the induction hypothesis to $M_{n-1}$. We get $n-1 = p-1$ and that the composition factors of the two series up to the $(n-1)^{\text{th}}$ place are the same. Since $M/M_{n-1} = M/K_{p-1}$, this is enough to complete the induction step.

Now suppose $M_{n-1} \neq K_{p-1}$. Then $\frac{M_{n-1}+K_{p-1}}{M_{n-1}}$ is a nonzero submodule of $M/M_{n-1}$ (it is an exercise to check that it can't be zero). Since $M/M_{n-1}$ is simple, we have $\frac{M_{n-1}+K_{p-1}}{M_{n-1}} = M/M_{n-1}$. We get

$$\frac{M_n}{M_{n-1}} = \frac{M_{n-1}+K_{p-1}}{M_{n-1}} \cong \frac{K_{p-1}}{M_{n-1} \cap K_{p-1}}$$

and similarly

$$\frac{K_p}{K_{p-1}} = \frac{M_{n-1}+K_{p-1}}{K_{p-1}} \cong \frac{M_{n-1}}{M_{n-1} \cap K_{p-1}}.$$

Now let

$$0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_t = M_{n-1} \cap K_{p-1}$$

be a composition series of $M_{n-1} \cap K_{p-1}$. This can be extended to a composition series for $M_{n-1}$ and to a composition series for $K_{p-1}$. The composition factors of the series

$$0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_t = M_{n-1} \cap K_{p-1} \subsetneq M_{n-1}$$

are $\{L_i/L_{i-1}\}$ and $M_{n-1}/M_{n-1} \cap K_{p-1}$, while the composition factors of

$$0 = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_t = M_{n-1} \cap K_{p-1} \subsetneq K_{p-1}$$

are $\{L_i/L_{i-1}\}$ and $K_{p-1}/M_{n-1} \cap K_{p-1}$. By the induction hypothesis, the composition factors of $K_{p-1}$ and $M_{n-1}$ are uniquely determined up to isomorphism. Therefore, the composition factors of

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

are $\{L_i/L_{i-1}\}$, $M_{n-1}/(M_{n-1} \cap K_{p-1})$ and $M/M_{n-1} \cong K_{p-1}/(M_{n-1} \cap K_{p-1})$, while the composition factors of the series

$$0 = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_{p-1} \subsetneq K_p = M$$

are $\{L_i/L_{i-1}\}$, $K_{p-1}/(M_{n-1} \cap K_{p-1})$ and $M/K_{p-1} \cong M_{n-1}/(M_{n-1} \cap K_{p-1})$. Thus, these two series have the same composition factors up to isomorphism, which proves the induction step. $\square$

The Jordan-Hölder Theorem shows that every module can be built up by taking iterated extensions of simple modules. It is also useful to understand the morphisms between simples. Over an algebraically closed field, this is very easy.

**Lemma 2.4** (Schur's Lemma)**.** *If $S_1$ and $S_2$ are simple modules over a finite-dimensional $\mathbb{C}$–algebra $A$ then*

$$\mathrm{Hom}_A(S_1, S_2) = \begin{cases} \mathbb{C} & \text{if} \quad S_1 \cong S_2 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Let $f : S_1 \to S_2$. Then if $f \neq 0$, then $\ker(f)$ is a proper submodule of $S_1$, so $\ker(f) = 0$. Also, $\mathrm{im}(f)$ is a nonzero submodule of $S_2$, so $\mathrm{im}(f) = S_2$. Therefore, $f$ is bijective, so is an isomorphism. If $g : S_1 \to S_2$ is another nonzero morphism, then $g^{-1}f : S_1 \to S_1$ is an isomorphism. Since we are working with finite-dimensional modules and our field is algebraically closed, this map must have an eigenvalue $\lambda$. Then the $\lambda$–eigenspace of $g^{-1}f$ is a nonzero submodule of $S_1$ and therefore equals $S_1$. So $f = \lambda g$. $\square$

## 3. Lecture 3

Another kind of module which can serve as a building block for all modules are the indecomposable modules. These are more general than simple modules, but the way in which every module is built up from indecomposables is much simpler.

3.1. **Indecomposable modules.**

**Definition 3.1.** *A module $M$ is called* indecomposable *if whenever $M = N_1 \oplus N_2$ for some submodules $N_1, N_2$, we have $N_1 = 0$ or $N_2 = 0$.*

Every simple module is indecomposable, but not-vice versa. For example, let $A = \mathbb{C}[t]/(t^2)$. Let $M = A$. Then $M$ is not simple because $tM$ is a proper nonzero submodule of $M$.

**Exercise 3.2.** *Show that $M$ is indecomposable.*

**Theorem 3.3** (Krull-Schmidt Theorem)**.** *Every module $M$ can be written*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

*where the $M_i$ are indecomposable. Furthermore, if*

$$M_1 \oplus M_2 \oplus \cdots \oplus M_n \cong N_1 \oplus N_2 \oplus \cdots \oplus N_p$$

*where the $N_i$ are indecomposable, then $n = p$ and the $M_i$ are isomorphic to the $N_j$, in some order.*

The existence part of the theorem is very easy to prove (do it now as an exercise). The uniqueness part is surprisingly difficult. We prove it using three lemmas, following the argument in [Bea99]. The first of these lemmas is very useful in its own right.

**Lemma 3.4** (Fitting's Lemma)**.** *A module $M$ is indecomposable if and only if every endomorphism of $M$ is either invertible or else nilpotent.*

*Proof.* If every endomorphism of $M$ is either a unit (another name for an invertible element) or nilpotent, then suppose $M = N_1 \oplus N_2$ with $N_1, N_2 \neq 0$. Then let

$$N_1 \xrightarrow{i_1} M \xrightarrow{\pi_1} N_1$$

$$N_2 \xrightarrow{i_2} M \xrightarrow{\pi_2} N_2$$

denote the inclusion and projection maps. Then $i_1\pi_1$ and $i_2\pi_2$ are both non-nilpotent endomorphisms of $M$, so they are invertible. But $i_1\pi_1 i_2\pi_2 = 0$, a contradiction.

Conversely, suppose $M$ is indecomposable. Let $f \in \mathrm{End}_A(M)$. Then

$$M \supset \mathrm{Im}(f) \supset \mathrm{Im}(f^2) \supset \cdots$$

is a descending chain of submodules of $M$. If $\mathrm{Im}(f^n) = 0$ for some $n$, then $f$ is nilpotent. If not, then the chain of submodules must still stabilise at some point, since $M$ is finite-dimensional. So there is some large enough $n$ so that

$$\mathrm{Im}(f^n) = \mathrm{Im}(f^{n+1}) = \mathrm{Im}(f^{n+2}) = \cdots$$

forever. In particular, $\mathrm{Im}(f^n) = \mathrm{Im}(f^{2n})$.

Now let $x \in M$. Then $f^n(x) = f^{2n}(z)$ for some $z$, so $x - f^n(z) \in \ker(f^n)$. Therefore, $x \in \mathrm{Im}(f^n) + \ker(f^n)$, and so $\mathrm{Im}(f^n) + \ker(f^n) = M$. By dimension, we must have $M = \mathrm{Im}(f^n) \oplus \ker(f^n)$. But $M$ is indecomposable, so either $\mathrm{Im}(f^n) = 0$ or $\mathrm{Im}(f^n) = M$. We are already assuming $f$ is not nilpotent, so $\mathrm{Im}(f^n) = M$, Therefore, $\det(f^n) \neq 0$ and so $\det(f) \neq 0$ and therefore $f$ is an isomorphism. □

**Remark 3.5.** There are two aspects of the above argument that we will use a lot. One is the *descending chain condition*. This says that if

$$M \supset M_1 \supset M_2 \supset M_3 \supset \cdots$$

is a descending chain of submodules of a module $M$, then there exists $n$ such that $M_n = M_{n+1} = \cdots$. This is true for any finite-dimensional module. In general, modules over a ring $R$ which satisfy the descending chain condition are called *Artinian*. We will often use Artinianness in our arguments, partly because it is helpful in dealing with more general situation, and partly because most of the arguments in this course are taken from books in which modules are not necessarily assumed to be finite-dimensional.

We also used the fact that $\det(f) \neq 0$ implies that $f$ is an isomorphism. Having access to ordinary linear algebra makes a lot of proofs a lot easier when dealing with finite-dimensional modules, and we will regularly use things like this.

**Lemma 3.6.** *If $M$ is an indecomposable module and $\lambda_i \in \mathrm{End}_A(M)$ and $\sum_{i=1}^n \lambda_i$ is invertible, then one of the $\lambda_i$ must be invertible.*

*Proof.* The proof is by induction on $n$. If $n = 1$ there is nothing to prove. If $u = \lambda_1 + \cdots + \lambda_n$ is invertible, then $u^{-1}\lambda_1 + \cdots + u^{-1}\lambda_n = 1$ and so $u^{-1}\lambda_2 + \cdots + u^{-1}\lambda_n = 1 - u^{-1}\lambda_1$. If $u^{-1}\lambda_1$ is a unit, then so is $\lambda_1$ and we are done. If not, then $u^{-1}\lambda_1$ is nilpotent and so $1 - u^{-1}\lambda_1$ is a unit, and we are done by induction. □

**Lemma 3.7.** *If $M, A, B$ are modules and $f : M \oplus A \to M \oplus B$ is an isomorphism and $\pi_M f i_M$ is an isomorphism, where $i_M : M \to M \oplus A$ is the inclusion and $\pi_M : M \oplus B \to M$ is the projection, then $A$ is isomorphic to $B$.*

*Proof.* Write elements of $M \oplus A$ as column vectors $\left(\begin{smallmatrix}\mu \\ \alpha\end{smallmatrix}\right)$ and similarly for $M \oplus B$. Then $f$ may be written as a matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $c : M \to B$ and $b : A \to M$. The hypothesis says that $a : M \to M$ is an isomorphism, and so is $f$. We wish to show that $d$ is an isomorphism. Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - ca^{-1}b \end{pmatrix}$$

The map $\left(\begin{smallmatrix} 1 & 0 \\ ca^{-1} & 1 \end{smallmatrix}\right)$ is an isomorphism; its inverse is $\left(\begin{smallmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{smallmatrix}\right)$. Therefore, $\left(\begin{smallmatrix} a & b \\ 0 & d-ca^{-1}b \end{smallmatrix}\right)$ is also an isomorphism. Regarding this as a block matrix and taking its determinant, we get $\det(a)\det(d-ca^{-1}b) \neq 0$ and therefore $\det(d-ca^{-1}b) \neq 0$. So $d - ca^{-1}b : A \to B$ is the desired isomorphism. □

3.2. **Proof of the Krull-Schmidt Theorem.** First, to show that every module is a finite sum of indecomposables, use induction on the length. If $\ell(M) = 1$ then $M$ is simple, so indecomposable. If $\ell(M) > 1$ then either $M$ is indecomposable or else $M = N_1 \oplus N_2$ with $N_1, N_2$ proper submodules. Then $\ell(N_1), \ell(N_2) < \ell(M)$ and we are done by induction.

Now suppose $M_i$, $N_i$ are indecomposable modules and

$$\varphi : M_1 \oplus \cdots \oplus M_k \to N_1 \oplus \cdots \oplus N_\ell$$

is an isomorphism. Let $i_r$ be the inclusion of $M_r$ into the direct sum of the $M_i$ and let $\pi_r$ be the projection from $\oplus M_i$ onto $M_r$. Similarly, let $j_r$ be the inclusion of $N_r$ into $\oplus N_i$ and let $p_r$ be the projection.

For $r = 1$, we have

$$id_{M_1} = \pi_1 i_1 = \pi_1 \left( \varphi^{-1} \sum_{r=1}^{\ell} j_r p_r \varphi \right) i_1 = \sum_{r=1}^{\ell} \pi_1 \varphi^{-1} j_r p_r \varphi i_1.$$

Since $M_1$ is indecomposable, we may apply Lemma 3.6 and conclude that one of the $\pi_1\varphi^{-1}j_r p_r \varphi i_1$ must be invertible. This implies that the short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{p_r \varphi i_1} N_r \longrightarrow \mathrm{cok}(p_r \varphi i_1) \longrightarrow 0$$

is split, a splitting map being $(\pi_1\varphi^{-1}j_r p_r \varphi i_1)^{-1} \pi_1 \varphi^{-1} j_r : N_r \to M_1$. We conclude that $N_r \cong M_1 \oplus \mathrm{cok}(p_r\varphi i_1)$. But $N_r$ is indecomposable and $M_1 \neq 0$, so $M_1 \cong N_r$. Relabel so that $r = 1$, then $M_1 \cong N_1$, an isomorphism being given by $p_1 \varphi i_1$. Lemma 3.7 now implies that $M_2 \oplus \cdots \oplus M_k \cong N_2 \oplus \cdots \oplus N_\ell$, and the proof may now be completed by induction on, say, $k$. (Yes, I *know* that $k$ also denotes the base field, and I really don't care.)

**Remark 3.8.** The Krull-Schmidt Theorem says that, to understand the entire module category, we only need to understand the indecomposable modules and the maps between them.

3.3. **Radicals.** Now we introduce the Jacobson radical, which is something from ring theory. We again follow the beautiful treatment of this subject in [Bea99]. First, we define a radical in general via two simple axioms.

In this section, $R$ can be *any* ring.

**Definition 3.9.** *A* radical *is a way of choosing, for every left $R$–module ${}_RM$, a submodule $\tau(M)$ satisfying the following two properties.*

- *If $f : M \to N$ then $f(\tau(M)) \subset \tau(N)$.*
- *For all $M$, $\tau(M/\tau(M)) = 0$.*

**Proposition 3.10.** *Basic properties:*

(1) *For all $M$, $\tau(R)M \subset \tau(M)$.*
(2) *$\tau(R)$ is a two-sided ideal of $R$.*

*Proof.* For the first part, let $m \in M$ and define $f : R \to M$ by $f(r) = rm$. Then $f(\tau(R)) = \tau(R)m \subset \tau(M)$. For the second part, $\tau(R)$ is automatically a left ideal because it is a left $R$–submodule of $R$ by definition. The first part also shows that it is a right ideal, if we take $M = R$. $\square$

**Proposition 3.11.** *Let $\mathcal{F}$ be a class of $R$–modules. Define*

$$\mathrm{rad}_{\mathcal{F}}(M) = \bigcap_{\substack{X \in \mathcal{F} \\ f : M \to X}} \ker(f).$$

*Then $\mathrm{rad}_{\mathcal{F}}$ is a radical. Furthermore, every radical is of the form $\mathrm{rad}_{\mathcal{F}}$ for some $\mathcal{F}$.*

*Proof.* It is an exercise to show that $\mathrm{rad}_{\mathcal{F}}$ is a radical.

Now suppose $\tau$ is a radical. We claim that $\tau = \mathrm{rad}_{\mathcal{F}}$ where $\mathcal{F} = \{X : \tau(X) = 0\}$. To see this, let $M$ be a module. If $x \in \tau(M)$ and $X \in \mathcal{F}$ and $f : M \to X$ then $f(x) \in \tau(X) = 0$ so $x \in \ker(f)$. Therefore, $\tau(M) \subset \mathrm{rad}_{\mathcal{F}}(M)$. Conversely, suppose $x \in \mathrm{rad}_{\mathcal{F}}(M)$. Take $f$ to be the quotient map $M \to M/\tau(M)$. Then $x \in \ker(f)$ and so $x \in \tau(M)$. So $\mathrm{rad}_{\mathcal{F}}(M) \subset \tau(M)$ as required. $\square$

## 4. Lecture 4

4.1. **The Jacobson radical.** The only radical we care about in this course is $\mathrm{rad}_{\mathcal{F}}$ where $\mathcal{F}$ is the class of all simple modules. This is called the *Jacobson radical*. It is the intersection of the kernels of all maps from $M$ to a simple module. Being the kernel of such a map is equivalent to being a maximal proper submodule (usually just called a maximal submodule). Therefore, we have the following definition.

**Definition 4.1.** *Let $M$ be an $R$–module. The* Jacobson radical *(or just* radical*) of $M$ is the submodule*

$$J(M) = \mathrm{rad}(M) = \bigcap(\textit{maximal submodules of } M).$$

By definition, the radical has to be a proper submodule unless $M = 0$. Also, the radical can be zero. This happens for example if 0 is a maximal submodule of $M$.

**Definition 4.2.** *An $R$–module $M$ is* semisimple *if* $\mathrm{rad}(M) = 0$.

**Proposition 4.3.** *If $M$ is a finite-dimensional module over an algebra $R$, then $M$ is semisimple $\iff$ $M$ is a direct sum of simple modules.*

*Proof.* If $M = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ with $S_i$ simple, then define $f_i : M \to S_i$ to be the projection onto $S_i$. Then $\mathrm{rad}(M) \subset \cap_{i=1}^n \ker(f_i) = 0$.

Conversely, if $\mathrm{rad}(M) = 0$ then we may observe that there are finitely many maps $f_1, f_2, \ldots, f_n$ from $M$ to a simple module with $\cap_{i=1}^n \ker(f_i) = 0$. Indeed, we can find a chain

$$\ker(f_1) \supsetneq \ker(f_1) \cap \ker(f_2) \supsetneq \ker(f_1) \cap \ker(f_2) \cap \ker(f_3) \supsetneq \cdots$$

which descends strictly at each step. Because $M$ is finite-dimensional, we get $\cap_{i=1}^n \ker(f_i) = 0$. Choose $f_1, \ldots, f_n$ such that this holds, and such that $n$ is as small as possible. In particular, $\ker(f_i) \neq M$ for all $i$. Suppose $f_i : M \to S_i$.

Now define

$$M \to S_1 \oplus S_2 \oplus \cdots \oplus S_n$$

by $m \mapsto (f_1(m), f_2(m), \ldots, f_n(m))$. This is injective because $\cap_i \ker(f_i) = 0$. To show that it is surjective, observe that for each $i$, $\ker(f_i) + \cap_{j \neq i} \ker(f_j) = M$. This is true because $\ker(f_i)$ is a maximal submodule and $\cap_{j \neq i} \ker(f_j) \neq 0$ by choice of $n$.

Now the proof can be completed by applying the *Chinese remainder theorem* which says that if $K_1, K_2, \ldots, K_n$ are submodules of a module $M$ and $K_i + \cap_{j \neq i} K_j = M$ for all $i$, then the natural map

$$\frac{M}{K_1 \cap K_2 \cap \cdots \cap K_n} \to \frac{M}{K_1} \oplus \frac{M}{K_2} \oplus \cdots \oplus \frac{M}{K_n}$$

is surjective. This can be proved by induction on $n$. $\square$

4.2. **Chracterisations of $J(R)$ for a ring $R$.** Regarding a ring $R$ as a left $R$–module, we have the following characterisations of the Jacobson radical $J(R) = \mathrm{rad}(R)$.

**Theorem 4.4.**

$$\begin{aligned}
J(R) &= \bigcap (\textit{maximal left ideals of } R) && (1)\\
&= \bigcap (\textit{left annihilators of simple left } R\textit{–modules}) && (2)\\
&= \{x \in R : 1 - ax \textit{ has a left inverse for all } a \in R\} && (3)\\
&= \textit{The largest two-sided ideal } I \textit{ of } R \textit{ such that } 1 - x \textit{ is a unit for all } x \in I && (4)
\end{aligned}$$

*Proof.* First, we show that (1) is equivalent to (2). The *left annihilator* of an $R$–module $M$ is $\mathrm{ann}(M) := \{x \in R : xm = 0 \text{ for all } m \in M\}$. It is a two-sided ideal of $R$. If $x$ belongs to the left annihilator of every simple left $R$–module, let $m$ be a maximal left ideal of $R$. Then $R/m$ is a simple left $R$–module and so $x \in \mathrm{ann}(R/m)$. Therefore, $x = x.1 \in m$, so $(2) \subset (1)$. Conversely, if $x$ is in every maximal left ideal, then $x \in J(R)$, which is a two-sided ideal by Proposition 3.10, so $xa \in J(R)$ for all $a \in R$. Now let $S$ be a simple $R$–module. Let $s \in S$, $s \neq 0$. Define $f : R \to S$ by $f(r) = rs$. Then $x \in \ker(f)$ so $xs = 0$. Furthermore, for every $a \in R$, $xa \in \ker(f)$ so $xas = 0$. Therefore, $xRs = 0$. But $Rs = S$ since $S$ is simple. So $x \in \mathrm{ann}(S)$. Therefore, $(1) = (2)$.

To show that (3) is equivalent to (1) or (2), first suppose that $x \in J(R)$ and $a \in R$. Then $ax \in J(R)$ and since $R(1 - ax) + Rax = R$, we get $R(1 - ax) + J(R) = R$. Write $M = R/R(1 - ax)$. Then $J(R)M = M$.

But by the axioms for a radical, $J(R)M \subset J(M)$, and so $M \subset J(M)$. We have already remarked that this is impossible unless $M = 0$. Therefore, $R = R(1 - ax)$. So there is some $b \in R$ with $1 = b(1 - ax)$. Therefore, $J(R) \subset (3)$. Conversely, suppose $x \in R$ and for all $a \in R$ there exists $b \in R$ with $b(1 - ax) = 1$. If $x \notin J(R)$ then there is a maximal left ideal $I$ of $R$ with $x \notin I$. Then $I + Rx = R$ and so there is $y \in I$ and $a \in R$ with $y + ax = 1$, so $y = 1 - ax \in I$. Therefore, $by = b(1 - ax) = 1 \in I$. So $I = R$ which contradicts that $I$ is supposed to be a proper ideal. Therefore, $(3) \subset J(R)$ and so $(3) = J(R)$.

To show that (4) is equivalent to the others, first we note that if $x \in J(R)$ then $x \in (3)$ and so $1 - x$ has a left inverse $b$. So $b(1 - x) = 1$. We want to show that $(1 - x)b = 1$. To see this, we have $b - bx = 1$ which implies $-bx = 1 - b$. Now, $-bx \in J(R)$ and so $1 - (1 - b)$ has a left inverse $c$, so $cb = 1$. Now

$$1 - x = cb(1 - x) = c \cdot 1 = c$$

and therefore $(1 - x)b = 1$. So $1 - x$ is a unit. This shows that $J(R)$ is an ideal of $R$ such that $1 - x$ is a unit for all $x \in J(R)$. We must now show that it contains every other two-sided ideal of $R$ with this property. To this end, let $I$ be a two-sided ideal of $R$ such that $1 - x$ is a unit for all $x \in I$. Then if $x \in I$, suppose there is a maximal left ideal $M$ of $R$ with $x \notin M$. Then $M + Rx = R$. So $1 = ax + m$ for some $a \in R$ and some $m \in M$. So $m = 1 - ax$. But $ax \in I$, so $1 - ax$ is a unit. Therefore, $m$ is a unit and $1 \in M$, a contradiction since $M$ is supposed to be a proper ideal. Therefore, $x \in M$ and so $I \subset J(R)$. This shows that (4) is well-defined and equals $J(R)$. $\square$

Note that (4) is left-right symmetric. Therefore, we could replace "left" by "right" in (1), (2) and (3) and the theorem would still be true. This gives us three more characterisations of the Jacobson radical for free.

In this course, the following proposition will be useful to us.

**Proposition 4.5.** *Let $A$ be a finite-dimensional algebra. Then the Jacobson radical of $A$ is the largest two-sided ideal $I$ of $A$ such that every $x \in I$ is nilpotent.*

*Proof.* If $I$ is an ideal of $A$ such that every $x \in I$ is nilpotent then let $x \in I$. Then $x^n = 0$ for some $n$ and so

$$(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 = (1 + x + x^2 + \cdots + x^{n-1})(1 - x).$$

So $1 - x$ is invertible for all $x \in I$. Therefore, $I \subset J(A)$ by (4) above.

Now, if $x \in J(A)$ then consider the following chain of left ideals of $A$

$$A \supset Ax \supset Ax^2 \supset \cdots$$

This must stabilise and so there exists an $n$ with $Ax^n = Ax^{n+1} = \cdots = Ax^{2n}$. Therefore, there is $a \in A$ with $x^n = ax^{2n}$, and so $(1 - ax^n)x^n = 0$. But $1 - ax^n$ is invertible since $ax^n \in J(R)$, so $x^n = 0$. $\square$

**Examples 4.6.** If $A$ is a commutative finite-dimensional algebra then $J(A) = \bigcap(\text{prime ideals of } A) = \{\text{nilpotent elements of } A\}$.

If $A$ is a finite-dimensional algebra and $M$ is an indecomposable $A$–module then $\mathrm{End}_A(M)$ is a finite-dimensional algebra, every element of which is either nilpotent or a unit (see Lemma 3.6). Using this, it is easy to see that the nilpotent elements form an ideal (thanks Shisen for pointing this out). This ideal must be the Jacobson radical.

4.3. **Projective modules.** Because every module has a projective resolution, it makes sense to look at projective modules. Recall the definition:

**Definition 4.7.** *A module $P$ is called* projective *if for all modules $M, N$ and surjections $M \to N \to 0$, if $f : P \to N$ then there exists a map $P \to M$ making the following diagram commute.*

$$\begin{tikzcd} & P \arrow[dl] \arrow[d, "f"] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Being projective is equivalent to being a direct summand of a free module. In the finite-dimensional case, all modules are finitely-generated and so $P$ is a projective $A$–module if and only if there is some $n$ with $A^{\oplus n} = P \oplus Q$ for some $Q$.

Not every module is projective in general. However, over a finite-dimensional algebra we can always find a "best approximation" to a given module by a projective module.

**Definition 4.8.** *If $M$ is an $A$–module, a* projective cover *of $M$ is a projective module $P$ together with a surjection $P \twoheadrightarrow M$ such that if $Q$ is any other projective module and $Q \twoheadrightarrow M$ is a surjection then there exists a* surjection *$Q \twoheadrightarrow P$ making the following diagram commute.*

$$\begin{tikzcd} & Q \arrow[dl, two heads] \arrow[d, two heads] \\ P \arrow[r, two heads] & M \end{tikzcd}$$

**Theorem 4.9.** *Every $A$–module has a projective cover.*

*Proof.* First, if $M$ is an $A$–module then if $m_1, \ldots, m_n$ is a basis of $M$ then $A^{\oplus n} \twoheadrightarrow M$ via $(a_1, \ldots, a_n) \mapsto \sum a_i m_i$. So there exists a projective $Q$ with a surjection $\pi : Q \twoheadrightarrow M$. Now let

$$\Omega = \{P \subset Q : \pi|_P \text{ is surjective}\}.$$

Let $P$ be an element of $\Omega$ of minimum length (we could also say: minimum dimension, but using length in such arguments is better form, because then they can more easily be generalised). We show that $P$ is a projective cover of $M$.

Consider the diagram

$$\begin{tikzcd} & Q \arrow[d, "\pi"] \\ P \arrow[r, two heads, "\pi|_P"'] & M \end{tikzcd}$$

Since $Q$ is projective, there exists $t : Q \to P$ making the diagram commute, so that $\pi|_P t = \pi$. Let $i : P \hookrightarrow Q$ be the inclusion. Then $\pi|_P ti = \pi|_P$. Therefore, $\operatorname{im}(ti) \subset P$ also surjects onto $M$. If $\operatorname{im}(ti)$ is a proper submodule of $P$, it has smaller length than $P$, which contradicts the choice of $P$. Therefore, $\operatorname{im}(ti) = P$. By finite-dimensionality, we get that $ti : P \to P$ is an isomorphism. The sequence

$$0 \longrightarrow P \xrightarrow{i} Q \longrightarrow Q/P \longrightarrow 0$$

is therefore split by the map $(ti)^{-1}t : Q \to P$. So $Q \cong P \oplus Q/P$ and therefore $P$ is projective.

To show that $P \twoheadrightarrow M$ is a projective cover, suppose $P'$ is a projective module and $\lambda : P' \twoheadrightarrow M$ is a surjection. Then by projectivity of $P'$, there exists $r : P' \to P$ with $\pi|_P r = \lambda$. Therefore, $r(P') \subset P \subset Q$ surjects onto $M$ under $\pi$, which implies that $r(P') = P$ since $P$ has minimum length. So $r$ is surjective, as required. □

**Proposition 4.10.** *If $P$ is a projective module then the quotient map $q : P \to P/\operatorname{rad}(P)$ is a projective cover.*

*Proof.* If $Q$ is projective and

$$\begin{tikzcd} & Q \arrow[d, two heads, "\lambda"] \\ P \arrow[r, "q"] & P/\operatorname{rad}(P) \end{tikzcd}$$

then there exists $t : Q \to P$ with $qt = \lambda$. Therefore, $t(Q) + \operatorname{rad}(P) = P$. Now if $t(Q) \neq P$ then there exists a maximal submodule $X \subsetneq P$ with $t(Q) \subset X$. But $\operatorname{rad}(P) \subset X$ because the radical is the intersection of the maximal submodules. So $t(Q) + \operatorname{rad}(P) \subset X \subsetneq P$, a contradiction. Therefore, $t(Q) = P$ and so $t$ is surjective, as required. □

**Proposition 4.11.** *Any two projective covers of a module are isomorphic, in the sense that if $P_1 \twoheadrightarrow M$ and $P_2 \twoheadrightarrow M$ are projective covers, then there exists an isomorphism $\lambda : P_1 \to P_2$ such that the following diagram commutes*

$$\begin{tikzcd} P_1 \arrow[dd, "\lambda"] \arrow[dr] & \\ & M \\ P_2 \arrow[ur] & \end{tikzcd}$$

*Proof.* By definition of a projective cover, there exists a surjection $\lambda : P_1 \to P_2$ making the diagram commute. There also exists a surjection $\mu : P_2 \to P_1$ making the diagram commute. Thus, $\mu\lambda : P_1 \to P_1$ is a surjection, hence an isomorphism by finite-dimensionality. This implies that $\lambda$ is injective, and therefore $\lambda$ is also an isomorphism. □

**Proposition 4.12.** *If $P$ is projective then $P$ is indecomposable $\iff$ $P/\mathrm{rad}(P)$ is simple.*

*Proof.* By the axioms for a radical, $P/\mathrm{rad}(P)$ is semisimple. So $P/\mathrm{rad}(P)$ is a direct sum of simples, by Proposition 4.3. If $P/\mathrm{rad}(P) = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ with $S_i$ simple, and $P_i \twoheadrightarrow S_i$ are projective covers, then it is an exercise to show that $\oplus_i P_i \to \oplus_i S_i$ is also a projective cover. By the previous two propositions, $P \cong \oplus_i P_i$, which implies that $n = 1$ since $P$ is assumed to be indecomposable.

Conversely, if $P = Q_1 \oplus Q_2$ then $\mathrm{rad}(P) = \mathrm{rad}(Q_1) \oplus \mathrm{rad}(Q_2)$ by the exercise below, and so $P/\mathrm{rad}(P) = Q_1/\mathrm{rad}(Q_1) \oplus Q_2/\mathrm{rad}(Q_2)$. Thus, $P/\mathrm{rad}(P)$ is not simple. □

**Exercise 4.13.** *Show that if $\tau$ is a radical then*

$$\tau(\oplus_{i=1}^n M_i) = \oplus_{i=1}^n \tau(M_i)$$

*for any modules $M_i$.*

Now we can use everything we have covered so far to prove a very nice theorem about the structure of a finite-dimensional algebra. This theorem holds over any base field $k$.

**Theorem 4.14.** *Let $A$ be a finite-dimensional $k$–algebra. Then there are finitely many isomorphism classes of simple $A$–modules $S_1, S_2, \ldots, S_n$. Let $P_i$ be the projective cover of $S_i$. Then $\{P_1, P_2, \ldots, P_n\}$ is a complete set of isomorphism classes of indecomposable projective $A$–modules. Furthermore, each $P_i$ is a summand of ${}_AA$ and every indecomposable summand of ${}_AA$ is isomorphic to one of the $P_i$.*

*Proof.* First, if $S$ is a simple $A$–module then we may define a map $A \to S$ by picking some nonzero $s \in S$ and sending $a \in A$ to $as$. This map must be surjective because $S$ is simple. This implies that $S$ is a composition factor of $A$. Therefore, by the Jordan-Hölder Theorem, there can be at most $\ell(A)$ distinct simple modules.

Call the simples $S_1, S_2, \ldots, S_n$. Let $P_i$ be the projective cover of $S_i$. Then $P_i$ is indecomposable. Indeed, if $P_i = Q_1 \oplus Q_2$ with $Q_1, Q_2 \neq 0$, then by simplicity of $S_i$, either $Q_1 \twoheadrightarrow S_i$ or $Q_2 \twoheadrightarrow S_i$. This is impossible because $P_i$ is a projective cover.

So $\{P_1, P_2, \ldots, P_n\}$ is a set of indecomposable projective $A$–modules. If $P$ is any other indecomposable projective $A$–module then $P \to P/\mathrm{rad}(P)$ is a projective cover by Proposition 4.10, and $P/\mathrm{rad}(P)$ is simple by Proposition 4.12, so must be isomorphic to one of the $S_i$. Therefore, $P \cong P_i$ by Proposition 4.11. Thus, $\{P_1, P_2, \ldots, P_n\}$ is a complete set of isomorphism classes of indecomposable projectives. Furthermore, no two modules in this set can be isomorphic, because if $S_i \ncong S_j$ then $P_i \ncong P_j$. Indeed, an indecomposable projective has a unique simple quotient because Proposition 4.12 shows that its radical has to be a maximal submodule, and is therefore the unique maximal submodule.

Next, if $P_i$ is one of the indecomposable projectives then because $A$ is projective and $A \twoheadrightarrow S_i$, we must have $A \twoheadrightarrow P_i$. There is an exact sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow P_i \longrightarrow 0,$$

which splits since $P_i$ is projective. So $P_i$ is a summand of $A$.

Finally, if $U$ is any indecomposable summand of $A$ then $U$ is projective and $U$ is a projective cover of $U/\mathrm{rad}(U)$ by Proposition 4.10. But $U/\mathrm{rad}(U)$ is simple by Proposition 4.12, so $U$ is the projective cover of one of the $S_i$ and hence is isomorphic to one of the $P_i$. □

**Definition 4.15.** *The indecomposable projectives $P_i$ are called* principal indecomposable modules*.*

We will show in the next lecture that each $P_i$ is of the form $Ae$ where $e \in A$ is an idempotent.

## 5. Lecture 5

At the end of the last lecture, we proved that every finite-dimensional algebra is of the form

$$A = P_1^{a_1} \oplus \cdots \oplus P_n^{a_n}$$

where the $P_i$ are the projective covers of the simple $A$–modules $S_i$. Note that each $P_i$ has a *unique* simple quotient $S_i$. This quotient is called the *head* or *top* of $P_i$. The rest of the composition factors of $P_i$ can be quite complicated, and people who study these things often devote effort to trying to calculate them.

**Exercise 5.1.** Let $k$ be a field and

$$A = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

be the algebra of $2 \times 2$ lower-triangular matrices over $k$. Find the simple $A$–modules and their projective covers.

*Tip: the correct way to approach this exercise is to think of yourself as a nineteenth-century biologist who has just been sent a specimen of some exotic animal in a brown paper parcel. It is no good trying to write down vector spaces with an $A$–action; this is like trying to understand the animal's habitat, about which you have no information. Instead, you want to look inside the animal. Get out your scalpel, dissect it, poke around in there and see what you can find. Indeed, if you can write down a composition series for $A$, you already know that every simple module will occur as one of the composition factors. You should not be afraid to get your hands dirty!*

**Exercise 5.2.** Repeat the previous exercise for the algebra $k[t]/(t^n)$.

5.1. **Semisimple algebras.** Now we consider a special class of algebras for which every module is projective.

**Theorem 5.3.** *Let $A$ be a finite-dimensional algebra. The following are equivalent.*

(1) $\mathrm{rad}(A) = 0$.

(2) *Every finite-dimensional $A$–module is projective.*

*Proof.* If $\mathrm{rad}(A) = 0$ then by Proposition 4.3, $A = S_1 \oplus \cdots \oplus S_n$, a direct sum of simple modules. Every simple $A$–module is a composition factor of $A$, and therefore every simple module appears as one of the $S_i$. So every simple $A$–module is projective. Now let $M$ be an arbitrary $A$–module. Then $M$ has a composition series

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M.$$

We have a short exact sequence

$$0 \longrightarrow M_{n-1} \longrightarrow M \longrightarrow M/M_{n-1} \longrightarrow 0$$

and this splits because $M/M_{n-1}$ is simple, hence projective. So $M \cong M_{n-1} \oplus M/M_{n-1}$. Continuing inductively yields

$$M \cong M_1 \oplus M_2/M_1 \oplus \cdots \oplus M/M_{n-1}$$

a direct sum of projectives. So $M$ is projective.

Conversely, if every $A$–module is projective, then in particular the composition factors of a composition series

$$0 = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n = A$$

are projective. As above, we obtain that $A$ is the direct sum of the $A_i/A_{i-1}$. So $A$ is a direct sum of simples, and therefore $\mathrm{rad}(A) = 0$ by Proposition 4.3. □

**Definition 5.4.** *An algebra $A$ is* semisimple *if* $\mathrm{rad}(A) = 0$.

*An abelian category $\mathcal{C}$ is* semisimple *if every object of $\mathcal{C}$ is projective (equivalently, every short exact sequence in $\mathcal{C}$ splits).*

Theorem 5.3 shows that $A$ is a semisimple algebra if and only if $A - \mathrm{mod}$ is a semisimple category. The following theorem is a structure theorem for semisimple algebras. The theorem is true over any algebraically closed field $\mathbb{C}$.

**Theorem 5.5** (Artin-Wedderburn)**.** *A finite-dimensional $\mathbb{C}$–algebra $A$ is semisimple if and only if there are some integers $n_i \geq 1$ with*

$$A \cong M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

*Proof.* First, we show that $B := M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ is semisimple. As a left module over itself, this algebra may be written as

$$(\mathbb{C}^{n_1})^{n_1} \oplus \cdots \oplus (\mathbb{C}^{n_r})^{n_r}$$

where $\mathbb{C}^{n_i}$ is the module of column vectors for $M_{n_i}(\mathbb{C})$. Each $\mathbb{C}^{n_i}$ is a simple $M_{n_i}(\mathbb{C})$–module, and hence also a simple module for the product $B$. Therefore, ${}_BB$ is a direct sum of simples, so is a semisimple algebra.

Conversely, suppose $A$ is a semisimple algebra. For any algebra $A$, we may define a map

$$A^{op} \to \mathrm{End}_A(A)$$

by $a \mapsto (r \mapsto ra)$. It is easy to check that this is a well-defined algebra map, and in fact an algebra isomorphism. So $A^{op} \cong \mathrm{End}_A(A)$. Now, since $A$ is semisimple, we may write $A = S_1^{a_1} \oplus S_2^{a_2} \oplus \cdots \oplus S_n^{a_n}$ where the $S_i$ are the simple modules. Here $S_i^{a_i}$ denotes the direct sum of $a_i$ copies of $S_i$, and we assume that $S_i \ncong S_j$ for $i \neq j$. We get

$$A^{op} \cong \mathrm{End}_A(\underbrace{S_1 \oplus \cdots \oplus S_1}_{a_1 \text{ copies}} \oplus \cdots \oplus \underbrace{S_n \oplus \cdots \oplus S_n}_{a_n \text{ copies}}).$$

In general, we may write homomorphisms $\oplus_i A_i \to \oplus_j B_j$ between two finite direct sums of modules as matrices with entries in the Hom-spaces $\mathrm{Hom}_A(A_i, B_j)$. We already saw this in the proof of the Krull-Schmidt Theorem above. Applying Schur's Lemma, we recall that

$$\mathrm{Hom}_A(S_i, S_j) = \begin{cases} \mathbb{C} & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\mathrm{End}_A(A)$ is isomorphic to the algebra of matrices of the form

$$\begin{bmatrix}
\mathbb{C} & \mathbb{C} & \cdots & \mathbb{C} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\mathbb{C} & \mathbb{C} & \cdots & \mathbb{C} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\mathbb{C} & \mathbb{C} & \cdots & \mathbb{C} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \mathbb{C} & \cdots & \mathbb{C} & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \vdots & \cdots & \vdots & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \mathbb{C} & \cdots & \mathbb{C} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \mathbb{C} & \cdots & \mathbb{C} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \mathbb{C} & \cdots & \mathbb{C}
\end{bmatrix}$$

with an $a_1 \times a_1$ block in the upper left-hand corner, followed by an $a_2 \times a_2$ block, etc. This algebra is clearly isomorphic to $M_{a_1}(\mathbb{C}) \times M_{a_2}(\mathbb{C}) \times \cdots \times M_{a_n}(\mathbb{C})$. To complete the proof, it suffices to show that $M_{a_1}(\mathbb{C}) \times M_{a_2}(\mathbb{C}) \times \cdots \times M_{a_n}(\mathbb{C}) \cong (M_{a_1}(\mathbb{C}) \times M_{a_2}(\mathbb{C}) \times \cdots \times M_{a_n}(\mathbb{C}))^{op}$. To see this, we can use two easy facts. First, for any algebras $A$ and $B$, $(A \times B)^{op} \cong A^{op} \times B^{op}$. This is left as an exercise. Second, $M_n(\mathbb{C})^{op} \cong M_n(\mathbb{C})$, an isomorphism being given by the map $A \mapsto A^T$ which takes a matrix to its transpose. $\square$

**Remark 5.6.** In general, if we work over a field that is not algebraically closed, then Schur's Lemma states that $\mathrm{Hom}(S_i, S_i)$ is a division ring (a ring in which every nonzero element is invertible), and the general Artin-Wedderburn Theorem states that if $R$ is an Artinian ring such that every $R$–module is projective, then $R$ is a product

$$M_{n_1}(\Delta_1) \times \cdots \times M_{n_r}(\Delta_r)$$

of matrix rings over division rings.

**Remark 5.7.** *Maschke's Theorem* states that if $G$ is a finite group, then the group algebra $\mathbb{C}G$ is a semisimple algebra. The Artin-Wedderburn Theorem then implies that $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, where the $n_i$ are the dimensions of the simple modules. This leads to the equation

$$|G| = \sum_{i=1}^{r} \dim(S_i)^2$$

which is useful in calculating the representations of finite groups.

**Historical Remark 5.8.** Wedderburn was a Scottish mathematician who was keen on canoeing and never married.

5.2. **What about injectives?** We have been concentrating on projective modules. Why have we neglected injectives? It turns out that there is a trick which enables us to get some information about the injectives for free.

**Definition 5.9.** *Define a functor*

$$D : A - \mathrm{mod} \to A^{op} - \mathrm{mod}$$

*by*

$$DM = \mathrm{Hom}_k(M, k) = M^*,$$

*the $k$–linear dual.*

The vector space $M^*$ is an $A^{op}$–module (a right $A$–module) because given $\phi : M \to k$ and $a \in A$ and $m \in M$, we may define $(\phi \cdot a)(m) = \phi(am)$. It is an exercise to check that this is well-defined and that $D$ is in fact a contravariant equivalence of categories. This works because $DD$ is the same as the identity functor. Applying Theorem 4.14 to $A^{op}$ and then applying $D$ yields the following theorem.

**Theorem 5.10.** *If $A$ is a finite-dimensional algebra then there are finitely many isomorphism classes of indecomposable injective modules $I_1, I_2, \ldots, I_n$, where $I_i$ is the injective envelope of the $i^{th}$ simple $S_i$.*

Recall that an *injective envelope* is the dual notion to a projective cover. The part of the theorem about the indecomposable projectives being summands of $A$ is of course not true for the injectives.

## 5.3. **Idempotents.**

**Definition 5.11.** *Let $R$ be a ring. An element $e \in R$ is called an* idempotent *if $e^2 = e$.*

Idempotents are rather useful in the study of finite-dimensional algebras.

Recall that $r \in R$ is *central* if $rs = sr$ for all $s \in R$.

**Definition 5.12.** *If $e, e_i$ are idempotents then*

(1) *$e_i$ and $e_j$ are called* orthogonal *if $e_i e_j = e_j e_i = 0$.*
(2) *$e \neq 0$ is* primitive *if whenever $e = e_1 + e_2$ with $e_1, e_2$ orthogonal idempotents, then either $e_1 = 0$ or $e_2 = 0$.*
(3) *$e \neq 0$ is a* primitive central idempotent *if $e$ is central and if $e = e_1 + e_2$ with $e_1, e_2$ central orthogonal idempotents then $e_1 = 0$ or $e_2 = 0$.*

5.4. **Central idempotents.** Let $A$ be a finite-dimensional algebra. Write $1 = f_1 + f_2 + \cdots + f_n$ where the $f_i$ are pairwise orthogonal central idempotents, and $n$ is as large as possible. Let us show that this uniquely determines the $f_i$. If $1 = f_1 + f_2 + \cdots + f_n = f_1' + f_2' + \cdots + f_m'$ then for each $i$, $f_i = f_i(f_1' + f_2' + \cdots + f_m') = f_i f_1' + \cdots + f_i f_m'$. If more than one term in this sum is nonzero, then we have $1 = \sum_{j \neq i} f_j + \sum f_i f_k'$, a sum of more than $n$ orthogonal central idempotents. Therefore, $f_i = f_i f_t'$ for some $t$. Similarly, $f_t' = f_t' f_j$ for some $j$, which forces $j = i$ since $f_i \neq 0$. Therefore, $f_t' = f_t' f_i = f_i f_t' = f_i$. This shows that each $f_i$ is equal to one of the $f_t'$, and conversely as well. So $\{f_1, \ldots, f_n\} = \{f_1', \ldots, f_m'\}$ as desired.

For any such decomposition of $1 \in A$ into a sum of orthogonal central idempotents $f_i$, we have

$$A = Af_1 \oplus Af_2 \oplus \cdots \oplus Af_n$$

as left $A$–modules. Each $Af_i$ is a two-sided ideal of $A$, and is an algebra in its own right with identity $f_i$, because $f_i(af_i) = af_i^2 = af_i$ for all $a \in A$. So *as algebras* we have

$$A \cong Af_1 \times Af_2 \times \cdots \times Af_n.$$

If $M$ is an $A$–module then for $m \in M$, we have $m = 1.m = f_1 m + \cdots + f_n m$. So

$$M = f_1 M \oplus f_2 M \oplus \cdots \oplus f_n M$$

and each $f_i M$ is an $A$–module because each $f_i$ is central. Furthermore, if $i \neq j$ then $\mathrm{Hom}_A(f_i M, f_j M) = 0$ because if $\psi : f_i M \to f_j M$ then $\psi(f_i M) = f_j \psi(f_i m) = \psi(f_i f_j m) = 0$. So each module decomposes into a direct sum of pieces, the $i^{th}$ piece being an $Af_i$ module, and such that there are no morphisms between the $i^{th}$ piece and the $j^{th}$ piece if $i \neq j$. This leads to a decomposition of the whole category of $A$–modules as

$$A - \mathrm{mod} \cong Af_1 - \mathrm{mod} \times \cdots \times Af_n - \mathrm{mod},$$

the product of categories.

**Definition 5.13.** *The categories $Af_i - \text{mod}$ are called the* blocks *of $A - \text{mod}$. The algebras $Af_i$ are called the* blocks *of $A$.*

In representation theory, people like to study the module category of an algebra"one block at a time".

**Example 5.14.** If

$$A = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{pmatrix}$$

then a decomposition of the identity element into pairwise orthogonal central idempotents is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we get $A \cong M_2(\mathbb{C}) \times \mathbb{C}$. We cannot decompose it further because although we can write $\text{diag}(1,1,0)$ as $\text{diag}(1,0,0) + \text{diag}(0,1,0)$, these two idempotents are not central. We will explain the meaning of non-central idempotents below.

**Remark 5.15.** As an aside, note that if $A \cong B \times C$ are algebras, then $1_A = (1_B, 0) + (0, 1_C)$ is a decomposition of $1_A$ into a sum of orthogonal central idempotents. Let us say an algebra $A$ is *not a product* if there are no algebras $A_1$ and $A_2$ with $A \cong A_1 \times A_2$. Then the above considerations show that the following theorem is true.

**Theorem 5.16** (Krull-Remak-Schmidt Theorem for algebras)**.** *If $A$ is a finite-dimensional algebra then $A \cong A_1 \times A_2 \times \cdots \times A_n$ where each $A_i$ is not a product, and if $A_1 \times \cdots \times A_n \cong B_1 \times \cdots \times B_m$ where each $B_i$ is not a product, then $m = n$ and $A_i \cong B_j$ in some order.*

*Proof.* We have shown the existence of such a decomposition above. For the uniqueness, we observe that the factors in the product are precisely the algebras $Af_i$, which depend only on $A$. □

5.5. **Non-central idempotents.** Now we consider what happens if you write $1 = e_1 + e_2 + \cdots e_N$ where the $e_i$ are pairwise orthogonal idempotents, but we do not insist that they are central. Clearly, $N \geq n$ where $n$ is the number of $f_i$ in the decomposition considered above. We show that each $e_i$ is primitive. Indeed, if $e_i = e_i' + e_i''$ where $e_i', e_i''$ are orthogonal idempotents, then $e_i' e_j = e_i'(e_i' + e_i'')e_j = e_i' e_i e_j = 0$ for all $j \neq i$. Similarly, $e_j e_i' = 0$ for all $j \neq i$, and the same for $e_i''$. So we get the decomposition $1 = \sum_{j \neq i} e_j + e_i' + e_i''$ of 1 into orthogonal idempotents, which contradicts the maximality of $N$.

**Claim 1.** *Each $Ae_i$ is an indecomposable $A$–module.*

To prove the claim, suppose $Ae_i = Q_1 \oplus Q_2$, where $Q_1$ and $Q_2$ are left $A$–modules. Then $e_i = q_1 + q_2$ for some $q_i \in Q_i$. Now,

$$e_i = e_i^2 = (q_1 + q_2)^2 = q_1^2 + q_2^2 + q_1 q_2 + q_2 q_1. \tag{1}$$

Also, $q_1 = q_1 e_i$ because $q_1 \in Ae_i$, and on the other hand $e_i = e_i^2 = e_i(q_1 + q_2) = e_i q_1 + e_i q_2 = q_1 + q_2$. Since $e_i q_1 \in Q_1$ and $e_i q_2 \in Q_2$, from $e_i q_1 + e_i q_2 = q_1 + q_2$ and the fact that $Q_1 \oplus Q_2$ is a direct sum, we get $e_i q_1 = q_1$. So $q_2 q_1 + q_1^2 = q_1$ while $q_1 = q_1 e_i$ implies $q_1 = q_1^2 + q_1 q_2$. Thus $q_1 q_2 = q_2 q_1 \in Q_1 \cap Q_2 = 0$. Therefore, going back to Equation (1), we have $e_1 = q_1^2 + q_2^2 = q_1 + q_2$. Thus, both of the $q_i$ are idempotents and they are orthogonal, which contradicts that $e_i$ is primitive. This proves the claim.

The decomposition $1 = \sum_{i=1}^N e_i$ gives

$$A = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_N.$$

We have just shown that the $Ae_i$ are indecomposable and therefore the Krull-Schmidt Theorem implies that $N$ and the modules $Ae_i$ are uniquely determined by $A$. Note that the $e_i$ themselves are *not* uniquely determined by $A$, for example in $M_2(\mathbb{C})$ we have the decompositions $1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. This contradicts what I said in the lecture; sorry!

We see that the $Ae_i$ are the indecomposable summands of $A$. In other words, they are precisely the indecomposable projectives.

**Example 5.17.** Let $A = M_3(\mathbb{C})$. Then we have the decomposition

$$1_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e_1 + e_2 + e_3$$

and each $Ae_i$ is isomorphic to the module $\mathbb{C}^3$ of column vectors. This shows that the $Ae_i$ can be isomorphic to each other.

**Example 5.18.** Now let us do Exercise 5.1. Recall that $A = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$. We may write the identity element as $e_1 + e_2$ where $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We get $A = Ae_1 \oplus Ae_2$, and $Ae_1 = \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right)$ while $Ae_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)$. We see that $S_2 := Ae_2$ is a simple module since it is one-dimensional, so it is its own projective cover. On the other hand, $Ae_1$ is not simple, since it has a submodule $M := \left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)$. This submodule is simple since it is one-dimensional; furthermore, it is a one-dimensional vector space on which an element $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ of $A$ acts as multiplication by $c$. Therefore, it is isomorphic to $S_2$. Thus, there is a composition series

$$0 \subsetneq M \cong S_2 \subsetneq Ae_1/M$$

of $Ae_1$. The module $S_1 := Ae_1/M$ is simple since it has dimension 1. It is the head of the indecomposable projective $Ae_1$. How do we know $Ae_1$ is indecomposable? One way is to observe that $M$ is a two-sided ideal of $A$ containing only nilpotent elements. Therefore, $M \subset \mathrm{rad}(A) \neq 0$. But if $Ae_1$ was decomposable, then $A$ would be a direct sum of three simple modules and we would have $\mathrm{rad}(A) = 0$, a contradiction.

Summary: the simple modules are $S_1 = \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right) / \left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)$ and $S_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)$. The projective cover of $S_2$ is $S_2$ and the projective cover of $S_1$ is $\left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right)$.

## 6. Lecture 6

6.1. **Quivers.** Quivers are really important (as we shall see) as well as being a source of computable examples. Colloquially, a quiver is a conical bag used for carrying arrows. In mathematics, a quiver means a finite directed graph. To be more precise, we make the following definition.

**Definition 6.1.** *A* quiver *is a 4–tuple $Q = (Q_0, Q_1, t, h)$ where*

- *$Q_0$ and $Q_1$ are finite sets.*
- *$t, h : Q_1 \to Q_0$ are functions.*

*We say that $Q_0$ is the set of* vertices *of $Q$, $Q_1$ is the set of* arrows *and $t(a), h(a)$ are the* tail *and* head *of the arrow $a$ respectively.*

We usually think of quivers pictorially. We draw a dot for each vertex in $Q_0$, and for each arrow $a$, we draw an arrow from the dot $t(a)$ to the dot $h(a)$. Loops at a vertex are allowed, and there can be any number of arrows between a given pair of vertices.

**Definition 6.2.** *Let $Q$ be a quiver. A* representation *$V = (\{V_i\}, \{\phi_a\})$ of $Q$ is*

- *A choice of a vector space $V_i$ for each $i \in Q_0$.*
- *A choice of a linear map $\phi_a : V_{t(a)} \to V_{h(a)}$ for each $a \in Q_1$.*
- *That's all! There is no other condition.*

**Examples 6.3.** (1) *Let*

$$Q = \bullet \longrightarrow \bullet$$

*Then some representations of $Q$ are:* $k \longrightarrow 0$ , $0 \longrightarrow k$ , $k \xrightarrow{\lambda} k$ , $\mathbb{C}^n \xrightarrow{A} \mathbb{C}^m$ , *where $A$ is some $m \times n$ matrix.*

(2) *Let $Q$ be the quiver $\circlearrowleft$ with one vertex and one loop. A representation of $Q$ is a choice of vector space $V$, together with a linear map $V \to V$.*

6.2. **Morphisms of representations.**

**Definition 6.4.** *If $Q$ is a quiver and $V = (\{V_i\}, \{\phi_a^V\}), W = (\{W_i\}, \{\phi_a^W\})$ are representations of $Q$, a* morphism *$\varphi : V \to W$ consists of a linear map $\varphi_i : V_i \to W_i$ for each $i \in Q_0$, such that for each arrow $a$, the following diagram commutes.*

$$\begin{tikzcd} V_{t(a)} \arrow[r, "\phi_a^V"] \arrow[d, "\varphi_{t(a)}"'] & V_{h(a)} \arrow[d, "\varphi_{h(a)}"] \\ W_{t(a)} \arrow[r, "\phi_a^W"] & W_{h(a)} \end{tikzcd}$$

You can see that identity morphisms and compositions are well-defined, so that representations of a given quiver $Q$ form a category. Furthermore, you can define direct sums, kernels and cokernels in this category, and it is possible to show that the category is abelian (we will show anyway that it is the category of modules over an algebra associated to $Q$).

**Examples 6.5.** (1) Let

$$Q = \bullet \longrightarrow \bullet$$

and let $\lambda, \mu \in k$ and let $W = k \xrightarrow{\lambda} k$ and $V = k \longrightarrow 0$ be representations of $Q$. Then

$$\begin{tikzcd} k \arrow[r, "\lambda"] \arrow[d, "\mu"] & k \arrow[d] \\ k \arrow[r] & 0 \end{tikzcd}$$

is a morphism $W \to V$. We see that $\mathrm{Hom}(W, V) = k$. On the other hand, $\mathrm{Hom}(V, W) = 0$.

(2) If $Q$ is the quiver with one vertex and one loop, and $V = (\mathbb{C}^n, A)$ is a representation of $Q$, then an automorphism $V \to V$ is the same thing as a matrix $g \in GL_n$ such that $g^{-1}Ag = A$. Classifying representations of $Q$ up to isomorphism is therefore the same as classifying matrices up to conjugacy. Over $\mathbb{C}$, this classification is given by the Jordan normal form, so we may identify the set of isomorphism classes of representations of $Q$ with the set $\bigsqcup_P \mathbb{C}^{|P|}$ where the union is over all partitions $P$, and $|P|$ denotes the number of parts of $P$.

**Definition 6.6.** *Let $Q$ be a quiver. A* path *in $Q$ is either a sequence of arrows $a_0a_1 \dots a_n$ such that $h(a_i) = t(a_{i-1})$ for all $i$, or one of the symbols $e_k$ for $k$ a vertex of $Q$. For a path $p = a_0a_1 \dots a_n$, we define $t(p) = t(a_n)$ and $h(p) = h(a_0)$, and we define $t(e_k) = h(e_k) = k$.*

The $e_k$ are called *trivial paths*. They are supposed to be paths which consist of starting at the vertex $k$, doing nothing, and staying at $k$. Given any two paths $p$ and $q$ with $h(q) = t(p)$, we have the *concatenation* $pq$ which is a well-defined path. If $q = e_{t(p)}$ then we define $pq = p$ and if $p = e_{h(q)}$ then we define $pq = q$.

**Definition 6.7.** *Let $Q$ be a quiver and $k$ a field. The* path algebra *of $Q$ is the vector space $kQ$ with basis given by the set of paths, and multiplication given by*

$$p \cdot q = \begin{cases} pq & \text{if } h(q) = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

It is an exercise to check that $kQ$ is a well-defined algebra. By definition, its dimension is equal to the number of paths in $Q$. Its unit element is given by $1 = \sum_{i \in Q_0} e_i$, as can be checked from the definition.

**Examples 6.8.** (1) If $Q$ is the quiver with one vertex and one loop $x$, then the paths in $Q$ are $e$ (the trivial path at the vertex) together with $x, x^2, x^3, \dots$. In this case, we have $kQ \cong k[x]$.

(2) If $Q$ is the quiver with one vertex and several loops $x_1, x_2, \ldots, x_n$, then $kQ$ is the free algebra $k\langle x_1, \ldots, x_n \rangle$.

(3) Let $Q = \bullet \longrightarrow \bullet$. Then the paths in $Q$ are $e_0, e_1$ and $a$, where 0 is the leftmost vertex and $a$ denotes the single arrow. The path algebra of $Q$ is the three-dimensional vector space $ke_0 + ke_1 + ka$ with multiplication given by $e_1 a = ae_0 = a$, $ae_1 = e_0 a = 0$, $e_0^2 = e_0$, $e_1^2 = e_1$, $a^2 = 0$, $e_0 e_1 = e_1 e_0 = 0$. This algebra is isomorphic to $\left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$ via $\left(\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right) \mapsto \beta a + \alpha e_0 + \gamma e_1$.

**Theorem 6.9.** *Let $Q$ be a quiver and $k$ a field. There is an equivalence of categories between the category of representations of $Q$ and the category of left $kQ$–modules.*

*Proof.* Let $V = (\{V_i\}_{i \in Q_0}, \{\phi_a\}_{a \in Q_1})$ be a representation of $Q$. Define a $kQ$–module $\overline{V}$ by $\overline{V} = \oplus_{i \in Q_0} V_i$, and if $p = a_0 a_1 \ldots a_n$ is a path and $v \in \overline{V}$, define

$$p \cdot v = \phi_{a_0} \phi_{a_1} \ldots \phi_{a_n}(v_{t(a_n)})$$

where $v = \sum_{i \in Q_0} v_i$. Also, define $e_k v = v_k$. It is necessary to check that $\overline{V}$ is really a $kQ$–module, and that $F : V \mapsto \overline{V}$ is a functor. This is left as an exercise.

To go the other way, if $X$ is a left $kQ$–module, define a representation $V = \widehat{X}$ of $Q$ by $V_i = e_i X$ for all $i \in Q_0$, and $\phi_a : e_{t(a)} X \to e_{h(a)} X$ by $\phi_a x = ax$. This time it is clear that $\widehat{X}$ is a well-defined representation of $Q$. We still need to check that $G : X \mapsto \widehat{X}$ is a functor.

Finally, it is necessary to check that $FG$ and $GF$ are naturally isomorphic to the respective identity functors. This is again left as an exercise. $\square$

**Example 6.10.** To see how the correspondence between representations and $kQ$–modules works in practice, consider the quiver $Q = \bullet \longrightarrow \bullet$ again. The path algebra of $Q$ is isomorphic to $\left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$. Let us work out which representation of $Q$ corresponds to the module $M = \left(\begin{smallmatrix} 0 & 0 \\ k & k \end{smallmatrix}\right)$. Using the fact that $e_0$ corresponds to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_1$ corresponds to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we get $e_0 M = 0$ and $e_1 M = M$. Therefore, $M$ is the representation

$$0 \longrightarrow k^2.$$

What about the representation $k \xrightarrow{1} k$ of $Q$? The corresponding $kQ$–module will be $k \oplus k = \left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right)$ with the action $e_0 \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right)$, $e_1 \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \beta \end{smallmatrix}\right)$ and $a \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}\right)$. In terms of lower triangular matrices, this is the module $P = \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right)$.

At the moment, we are mostly interested in finite-dimensional path algebras.

**Lemma 6.11.** *Let $Q$ be a quiver and let $k$ be a field. Then $kQ$ is finite-dimensional if and only if $Q$ has no oriented cycles (including loops).*

*Proof.* If $Q$ has an oriented cycle $c$, then $c, c^2, c^3, \ldots$ are basis elements of $kQ$. Conversely, if $Q$ has no oriented cycles, then no path in $Q$ can be longer than the number of edges in $Q$, so there is an upper bound for the lengths of the paths in $Q$ and therefore there are only a finite number of paths. $\square$

We will consider only those quivers $Q$ with no cycles. We will calculate the simple, indecomposable projective and indecomposable injective modules for such a quiver. These things turn out to be easy to calculate, which means that quivers are a good source of examples. Furthermore, they are very important for theoretical reasons, as will be shown below.

## 7. Lecture 7

Let $Q$ be a quiver with no oriented cycles, $k$ a field, and $kQ$ the path algebra of $Q$. Recall that we have

$$1 = \sum_{i \in Q_0} e_i$$

and that the $e_i$ are pairwise orthogonal idempotents. Thus, we have the decomposition

$$A = \bigoplus_{i \in Q_0} Ae_i.$$

We claim that each $Ae_i$ is indecomposable. To see this, we use the following proposition.

**Proposition 7.1.**

$$\mathrm{Hom}_A(Ae_i, Ae_j) \cong e_i A e_j$$

*as vector spaces.*

To prove the proposition, map $f \in \mathrm{Hom}_A(Ae_i, Ae_j)$ to $f(e_i)$. It is an exercise to show that this gives a bijection.

In particular, $\mathrm{End}_A(Ae_i) \cong e_i A e_i$. This space has a basis given by the paths from the vertex $i$ to the vertex $i$. Since $Q$ has no oriented cycles, we must have $e_i A e_i = k e_i$. Therefore, $\mathrm{End}_A(Ae_i) \cong k$. Now Fitting's Lemma implies that $Ae_i$ is an indecomposable module. The Krull-Schmidt Theorem then says that $\{Ae_i : i \in Q_0\}$ is the complete set of projective indecomposable $A$–modules.

Now we calculate the simple modules. We have a vector space decomposition $Ae_i = \oplus_j e_j A e_i$. Because there is no nontrivial path from $i$ to $i$ for any vertex $i$, we see that $\oplus_{j \neq i} e_j A e_i$ is a submodule, and it has codimension 1. Therefore, it is a maximal submodule. Since the radical of an indecomposable projective module is a maximal submodule, and the radical is by definition the intersection of all the maximal submodules, it follows that an indecomposable projective has a unique maximal submodule and that this is equal to the radical. Therefore, $\mathrm{rad}(Ae_i) = \bigoplus_{j \neq i} e_j A e_i$, and the quotient

$$S_i = Ae_i / \mathrm{rad}(Ae_i)$$

is the head of $Ae_i$. As a representation of $Q$, it has a $k$ at the vertex $i$ and zeroes at all the other vertices. Clearly, if $i \neq j$ then $S_i$ is not isomorphic to $S_j$, since $\mathrm{Hom}_A(S_i, S_j) = 0$.

**Examples 7.2.** Consider first the quiver $Q = \bullet \longrightarrow \bullet$ as before. Label the vertices $0 \longrightarrow 1$ The simple $kQ$–modules are $S_0 = k \longrightarrow 0$ and $S_1 = 0 \longrightarrow k$ . The projective cover of $S_1$ is $Ae_1$, the span of all the paths starting at the vertex 1. This is just $S_1$ itself, so $S_1$ is projective. The projective cover of $S_0$ is $Ae_0$. Viewed as a representation of $Q$, this is the representation whose vector space at the vertex $i$ is the span of the paths from 0 to $i$. We may write it as $k \xrightarrow{1} k$ .

The same pattern holds in general. Here is a more complicated example. Let us consider the quiver

$$Q = \begin{tikzcd} & & \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] & \bullet \arrow[ur, shift left] \arrow[ur, shift right] & & \\ & & \bullet \arrow[ul] & \bullet \arrow[l] \end{tikzcd}$$

with the vertices labelled as follows.

$$\begin{tikzcd} & & 1 \arrow[r] & 2 \\ 0 \arrow[r] & 3 \arrow[ur, shift left] \arrow[ur, shift right] & & \\ & & 4 \arrow[ul] & 5 \arrow[l] \end{tikzcd}$$

There is one projective simple module $S_2$, given by

$$\begin{tikzcd} & & 0 \arrow[r] & k \\ 0 \arrow[r] & 0 \arrow[ur, shift left] \arrow[ur, shift right] & & \\ & & 0 \arrow[ul] & 0 \arrow[l] \end{tikzcd}$$

None of the other simples are projective, because for every vertex $i \neq 2$, there is a nontrivial path from $i$ to some other vertex. Some examples of simple modules $S_i$ and their projective covers $P_i$ are:

$$S_0 = \begin{tikzcd} & & 0 \arrow[r] & 0 \\ k \arrow[r] & 0 \arrow[ur, shift left] \arrow[ur, shift right] & & \\ & & 0 \arrow[ul] & 0 \arrow[l] \end{tikzcd}$$

with projective cover

$$\begin{tikzcd}
& & k^2 \arrow[r, "id"] & k^2 \\
k \arrow[r] & k \arrow[ur, shift left] \arrow[ur, shift right] & & \\
& & 0 \arrow[ul] & 0 \arrow[l]
\end{tikzcd}$$

where the two maps $k \to k^2$ are the canonical insertions of $k$ into $k \oplus k$, and

$$S_4 = \begin{tikzcd}
& & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[ur, shift left] \arrow[ur, shift right] & & \\
& & k \arrow[ul] & 0 \arrow[l]
\end{tikzcd}$$

with projective cover:

$$\begin{tikzcd}
& & k^2 \arrow[r, "id"] & k^2 \\
0 \arrow[r] & k \arrow[ur, shift left] \arrow[ur, shift right] & & \\
& & k \arrow[ul] & 0. \arrow[l]
\end{tikzcd}$$

How about injective modules? We can get these by finding the indecomposables projectives for $(kQ)^{op}$ and then taking their linear duals.

**Lemma 7.3.** *If $Q = (Q_0, Q_1, t, h)$ is a quiver, define the* opposite quiver $Q^{op} := (Q_0, Q_1, h, t)$. *That is, $Q^{op}$ is $Q$ with all arrows reversed. Then*

$$(kQ)^{op} \cong kQ^{op}.$$

*Proof.* Exercise. □

**Example 7.4.** Here is an example. Let $Q$ be the quiver

$$Q = \bullet \longrightarrow \bullet \longleftarrow \bullet$$

with the vertices labelled $1, 2, 3$ from left to right. Here is a table of the simple modules $S_i$ and their projective covers $P_i$.

$$
\begin{array}{llll}
S_1 = & k \longrightarrow 0 \longleftarrow 0 & P_1 = & k \xrightarrow{1} k \longleftarrow 0 \\
S_2 = & 0 \longrightarrow k \longleftarrow 0 & P_2 = & 0 \longrightarrow k \longleftarrow 0 \\
S_3 = & 0 \longrightarrow 0 \longleftarrow k & P_3 = & 0 \longrightarrow k \xleftarrow{1} k
\end{array}
$$

To find the indecomposable injectives, we should take the indecomposable projectives for the opposite quiver

$$Q^{op} = \bullet \longleftarrow \bullet \longrightarrow \bullet$$

and dualise them. The simple $Q^{op}$ modules and their projective covers are

$$
\begin{array}{llll}
\Sigma_1 = & k \longleftarrow 0 \longrightarrow 0 & \Pi_1 = & k \longleftarrow 0 \longrightarrow 0 \\
\Sigma_2 = & 0 \longleftarrow k \longrightarrow 0 & \Pi_2 = & k \xleftarrow{1} k \xrightarrow{1} k \\
\Sigma_1 = & 0 \longleftarrow 0 \longrightarrow k & \Pi_1 = & 0 \longleftarrow 0 \longrightarrow k
\end{array}
$$

and you will find that taking the duals of the $\Pi_i$ gives the indecomposable injectives for $Q$.

$$
\begin{array}{ll}
I_1 = & k \longrightarrow 0 \longleftarrow 0 \\
I_2 = & k \xrightarrow{1} k \xleftarrow{1} k \\
I_3 = & 0 \longrightarrow 0 \longleftarrow k
\end{array}
$$

7.1. **Projective dimension.** Now we need to recall some homological algebra.

**Definition 7.5.** *A* projective resolution *of a module $M$ is an exact sequence*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

*where each $P_i$ is projective. We say that the resolution has* length $n$ *if $P_{n+1} = P_{n+2} = \cdots = 0$.*

**Definition 7.6.** *Define the* projective dimension $\mathrm{pd}M \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ *to be the minimum length of a projective resolution of $M$.*

*If $A$ is an algebra, define the* global dimension *(aka* homological dimension*) of $A$ to be*

$$\mathrm{gldim}(A) = \sup\{\mathrm{pd}M : M \in A - \mathrm{mod}\}.$$

We need to use Ext, so let us recall its basic properties.

First, if $A$ is a $k$–algebra then for any $A$–modules $M, N$ and any $n \geq 0$, $\mathrm{Ext}^n_A(M, N)$ is a $k$–vector space which can be thought of as a kind of generalised Hom. Indeed, $\mathrm{Ext}^0_A(M, N) = \mathrm{Hom}_A(M, N)$. In general, we define $\mathrm{Ext}^n_A(M, N)$ as follows. Let

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be a projective resolution of $M$. Write the maps in this resolution as $d_n : P_n \to P_{n-1}$. Take Hom into $N$, and chop off the $\mathrm{Hom}(M, N)$ term to get a complex of vector spaces which goes to the right:

$$0 \longrightarrow \mathrm{Hom}_A(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_A(P_1, N) \xrightarrow{d_2^*} \cdots \xrightarrow{d_n^*} \mathrm{Hom}_A(P_n, N) \xrightarrow{d_{n+1}^*} \cdots$$

Define

$$\mathrm{Ext}_A(M, N) := \ker(d_{n+1}^*)/\mathrm{im}(d_n^*).$$

One of the most important properties of the Ext groups is the long exact sequence in cohomology. If

$$0 \to M'' \to M \to M' \to 0$$

is a short exact sequence of $A$–modules and $N$ is an $A$–module, then the sequence

$$0 \to \mathrm{Hom}_A(M', N) \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M'', N)$$

is always exact. In fact, this can be extended to an exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M', N) \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M'', N) \to \mathrm{Ext}_A^1(M', N) \to \mathrm{Ext}_A^1(M, N) \to \cdots \\ \mathrm{Ext}_A^1(M'', N) \to \mathrm{Ext}_A^2(M', N) \to \mathrm{Ext}_A^2(M, N) \to \mathrm{Ext}_A^2(M'', N) \to \mathrm{Ext}_A^3(M', N) \to \cdots \end{aligned}$$

and so on. Although the maps in this sequence can be written down explicitly, it is often just the existence of such an exact sequence which is useful in applications.

**Fact 7.7.** *If $A$ is an algebra then*

$$\mathrm{gldim}(A) \leq n \iff \mathrm{Ext}_A^{n+1} = 0.$$

*Proof.* If every module has a projective resolution of length $\leq n$ then a direct calculation shows that $\mathrm{Ext}^{n+1} = 0$.

Conversely, if

$$P_n \xrightarrow{\theta_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\theta_0} M \longrightarrow 0$$

is a projective resolution of a module $M$ then let $Q_i = P_{n+i}$ and consider the following sequence, which is a projective resolution of $\mathrm{im}(\theta_n)$.

$$Q_n \xrightarrow{\theta_{2n}} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{\theta_n} \mathrm{im}(\theta_n) \longrightarrow 0$$

If $N$ is any module, we get that $\mathrm{Ext}^1(\mathrm{im}(\theta_n), N) = \mathrm{Ext}^{n+1}(M, N) = 0$ by hypothesis. So $\mathrm{im}(\theta_n)$ is projective and we obtain the following resolution of $M$ of length $\leq n$

$$0 \longrightarrow \mathrm{im}(\theta_n) \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow M \longrightarrow 0$$

□

Note that the fact that $\mathrm{Ext}^1(K,-)=0$ implies that $K$ is projective, which was used in the above proof, can be deduced from the long exact sequence for Ext. For details of the above, see the book [Rot09].

**Definition 7.8.** *Let $A$ be an algebra. $A$ is said to be* heriditary *if* $\mathrm{gldim}(A) \leq 1$.

We've already dealt with the case $\mathrm{gldim}(A) = 0$ for a finite-dimensional $\mathbb{C}$–algebra $A$. This was the Artin-Wedderburn Theorem. Our aim now is to deal with the case $\mathrm{gldim}(A) \leq 1$. We now explain why this is called "hereditary".

**Fact 7.9.** $\mathrm{gldim}(A) \leq 1 \iff$ *every submodule of a projective module is projective.*

*Proof.* If every submodule of a projective is projective, then if $M$ is a module, we can first find a projective $P_0$ which surjects onto $M$, and then we have an exact sequence

$$0 \to K \to P_0 \to M \to 0$$

where $K$ is the kernel. Then $K$ must be projective, so $M$ has a projective resolution of length $\leq 1$. Therefore, the global dimension of $A$ does not exceed 1.

Conversely, if $\mathrm{gldim}(A) \leq 1$, then $\mathrm{Ext}^2_A = 0$. Suppose $P$ is a projective $A$–module and $K \subset P$. Then

$$0 \to K \to P \to P/K \to 0$$

is an exact sequence. Let $N$ be any module and apply the long exact sequence:

$$0 \to \mathrm{Hom}(P/K,N) \to \mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N) \to \mathrm{Ext}^1(P/K,N) \to \cdots$$
$$\mathrm{Ext}^1(P,N) \to \mathrm{Ext}^1(K,N) \to \mathrm{Ext}^2(P/K,N) = 0$$

Since $P$ is projective, $\mathrm{Ext}^1(P,-) = 0$ and therefore $\mathrm{Ext}^1(K,N) = 0$ for all $N$, which implies that $K$ is projective. □

The reason for the name "hereditary" is that a submodule of a projective module *inherits* the property of being projective.

Our next aim is to show that path algebras are hereditary. Ultimately, we will show that if $A$ is a hereditary finite-dimensional algebra then $A-\mathrm{mod}$ is equivalent to $kQ-\mathrm{mod}$ where $Q$ is some quiver.

## 8. Lecture 8

In this lecture we will prove that path algebras are hereditary in two ways. The first way is to directly write down a resolution of an arbitrary module $M$. It works for any quiver, even one with loops. The argument is copied directly from [CB].

**Theorem 8.1.** *Let $Q$ be a quiver and $M$ a $kQ$–module. Then there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{a\in Q_1} Ae_{h(a)} \otimes_k e_{t(a)}M \xrightarrow{f} \bigoplus_{i\in Q_0} Ae_i \otimes_k e_iM \xrightarrow{g} M \longrightarrow 0$$

*where the map $g$ is given on the $e_i$–component by $g(\alpha \otimes x) = \alpha x$ and the map $f$ is given on the $a$–component of the direct sum by $f(\gamma \otimes x) = \gamma a \otimes x - \gamma \otimes ax$.*

*Proof.* The map $g$ is onto because $M = \sum_{i \in Q_0} kQe_iM$. A direct calculation shows that $gf = 0$. The harder parts are to show that $f$ is injective and to show that $\ker(g) \subset \operatorname{im}(f)$.

To show that $\ker(f) = 0$, let $\xi \in \ker(f)$ and write

$$\xi = \sum_{a \in Q_1} \sum_p p \otimes x_{p,a}$$

where $x_{p,a}$ is some element of $M$, and the second summation is over all paths $p$ with $h(p) = a$. Suppose $\xi \neq 0$. Then let $p$ be a path of maximal length such that $x_{p,a} \neq 0$. Then

$$f(\xi) = \sum_a \sum_p pa \otimes x_{p,a} - \sum_a \sum_p p \otimes ax_{p,a}.$$

Because $p$ was chosen of maximal length, the only term of $f(\xi)$ in which $pa$ appears in the first factor is $pa \otimes x_{p,a}$. Since $x_{p,a}$ was taken to be nonzero, we get $f(\xi) \neq 0$, a contradiction. Thus, $\xi = 0$.

To show that $\ker(g) \subset \operatorname{im}(f)$, let

$$\zeta = \sum_p p \otimes m_p$$

be an element of $\ker(g)$, where the sum is over the paths $p$ in $Q$. If $p$ is a path of length $> 0$ (ie. a nontrivial path) then $p = p'a$ for some path $p'$ and some arrow $a$. Then $f(p' \otimes m_p) = p \otimes m_p - p' \otimes am_p$. So $\zeta - f(p' \otimes m_p)$ has no $p$–component. Continuing inductively, we can find an element $w \in \operatorname{im}(f)$ with

$$\zeta - w = \sum e_i \otimes m_i$$

for some $m_i \in e_iM$. Now, $g(\zeta - w) = g(\zeta) - g(w) = 0$ because we have already shown that $gf = 0$. Thus, $\sum m_i = 0 \in \bigoplus e_iM$. This implies $m_i = 0$ for all $i$ and so $\zeta - w = 0$, so $\zeta \in \operatorname{im}(f)$. $\square$

**Remarks 1.** It is worth going over the above proof carefully as it is difficult to understand such a proof during the lecture. It was copied almost word-for-word from [CB], so it is worth looking at those as well.

**Corollary 8.2.** *For any quiver $Q$, $kQ$ is a hereditary algebra.*

*Proof.* It suffices to note that the modules $\bigoplus_{a \in Q_1} Ae_{h(a)} \otimes_k e_{t(a)}M$ and $\bigoplus_{i \in Q_0} Ae_i \otimes_k e_iM$ are projective for any $M$, because as left $A$–modules they are isomorphic to direct sums of copies of $Ae_r$ for various $r$. $\square$

In the case of a quiver with no loops, we now give another proof that path algebras are hereditary. It is useful to give both proofs, because there are aspects of both which will be useful to us.

**Theorem 8.3.** *Let $A$ be a finite-dimensional algebra with Jacobson radical $J$. Then*

$$\operatorname{gldim}(A) = \operatorname{pd}_A(A/J).$$

*Proof.* Write $A = \bigoplus P_i^{a_i}$ as a direct sum of indecomposable projectives. Because radicals commute with finite direct sums, $J = \mathrm{rad}(A) = \bigoplus \mathrm{rad}(P_i)^{a_i}$. Therefore, $A/J = \bigoplus (P_i/\mathrm{rad}(P_i))^{a_i}$. This shows that every simple $A$–module is a summand of the $A$–module $A/J$. Therefore, if $\mathrm{pd}(A/J) \leq n$ then $\mathrm{pd}(S) \leq n$ for every simple $A$–module $S$.

(Here we use the fact that for any modules $M_i$, $\mathrm{pd}(M_1 \oplus \cdots M_n) = \max\{\mathrm{pd}(M_1), \ldots, \mathrm{pd}(M_n)\}$. This can be proved by observing that $\mathrm{pd}(M)$ is the smallest nonnegative integer $n$ such that $\mathrm{Ext}^{n+1}(M, -) = 0$ and using the fact that Ext functors commute with finite direct sums).

Now if $\mathrm{pd}(S) \leq n$ for all simple $S$, then $\mathrm{pd}(M) \leq n$ for all $M$, because the *Horseshoe lemma* implies that if

$$0 \to M'' \to M \to M' \to 0$$

is an exact sequence, then $\mathrm{pd}(M) \leq \max\{\mathrm{pd}(M'), \mathrm{pd}(M'')\}$. For the Horseshoe Lemma, see [Rot09]. Using this fact, together with the fact that every module has a composition series, we can show by induction on the length of $M$ that $\mathrm{pd}(M) \leq n$ for all $M$. □

**Corollary 8.4.** *A finite-dimensional algebra $A$ is hereditary if and only if the Jacobson radical $J$ is a projective $A$–module.*

*Proof.* If $J$ is a projective $A$–module then $0 \to J \to A \to A/J \to 0$ is a projective resolution of $A/J$ and therefore $\mathrm{pd}(A/J) \leq 1$. Conversely, if $A$ is hereditary then $J \subset A$ is a projective module because it is a submodule of a projective. □

**Definition 8.5.** *If $Q$ is a quiver then the* arrow ideal *of $kQ$ is the two-sided ideal of $kQ$ generated by all the arrows.*

**Proposition 8.6.** *If $Q$ is a quiver without oriented cycles then the Jacbson radical $J$ of $kQ$ is equal to the arrow ideal of $kQ$.*

*Proof.* Let $A$ be the arrow ideal of $KQ$. If $a$ is an arrow of $kQ$ then $(1-a)(1+a) = (1+a)(1-a) = 1$ so $1-a$ is a unit and therefore $a \in J$. Therefore, $A \subset J$. Conversely, suppose $I$ is a two-sided ideal of $kQ$ such that every $x \in I$ is nilpotent. Suppose $x := \sum \alpha_i e_i + \sum_p \beta_p p \in I$ where the second sum is over all paths of length $> 0$. We want to show that each $\alpha_i = 0$. If not, then $e_i x e_i = \alpha_i e_i \in I$ and so $e_i \in I$. But $e_i$ is not nilpotent. Therefore, $\alpha_i = 0$ for all $i$, and $I \subset A$. In particular, $J \subset A$ and therefore $A = J$ as required. □

Note that I mistakenly said in the lecture that the arrow ideal is equal to the Jacobson radical for any path algebra. This is completely untrue, for example for the quiver with one vertex and one loop, the path algebra is $k[x]$ and the arrow ideal is $(x)$, but the Jacobson radical is 0.

Now we can show that any finite-dimensional path algebra $A$ is hereditary in the following way. We just need to show that the Jacobson radical is a projective left $A$–module. The Jacobson radical may be written

$$J = \bigoplus_{a \in Q_1} Aa = \bigoplus_{a \in Q_1} Ae_{h(a)}a.$$

as a left $A$–module, this is isomorphic to $\bigoplus_{a \in Q_1} Ae_{h(a)}$. This is a projective module, which proves that $A$ is hereditary.

It is interesting that path algebras are hereditary, but even more interesting is the fact that the path algebras of cycle-free quivers are the only examples of hereditary algebras up to Morita equivalence (a notion to be defined later). Our next aim is to prove a more general theorem than this. We will show that if $A$ is a finite-dimensional algebra then $A-\text{mod}$ is equivalent to $kQ/I-\text{mod}$ for some quiver $Q$ and some ideal $I$ of $kQ$. We will then show that the hereditary algebras are exactly those for which $I = 0$.

8.1. **Elementary algebras.** Our first aim is to associate a quiver to a finite-dimensional algebra $A$. We restrict for the moment to so-called *elementary algebras*.

**Theorem 8.7.** *If $A$ is a finite-dimensional algebra over an algebraically closed field $k$ and $J$ is the Jacobson radical of $A$ then $A/J$ is semisimple.*

*Proof.* By the axioms for a radical, $\text{rad}(A/J) = 0$ and therefore $A/J$ is a direct sum of simple $A$–modules by Proposition 5.3. Each of these modules is also an $A/J$–module, and furthermore they are simple $A/J$–modules. Therefore, $A/J$ is a direct sum of simple $A/J$–modules, so is a semisimple algebra. □

The Artin-Wedderburn Theorem now says that $A/J$ is a direct product of matrix rings over $k$. If all of these matrix rings are $1 \times 1$, then $A$ is said to be elementary.

**Definition 8.8.** *Let $k$ be an algebraically closed field and $A$ a finite-dimensional $k$–algebra. Then $A$ is called* elementary *if*

$$A/J \cong k \times k \times \cdots \times k$$

*where $J$ is the Jacobson radical of $A$.*

For example, if $Q$ is a quiver without loops then $\mathbb{C}Q/J \cong \times_{i \in Q_0} \mathbb{C}e_i$, so $\mathbb{C}Q$ is elementary. On the other hand, $M_2(\mathbb{C})$ is not elementary.

## 9. Lecture 9

Recap: if $A$ is a finite-dimensional algebra and $J$ is the Jacobson radical of $A$ ($J = \text{rad}(A)$, but we usually write it as $J$ when we are thinking of it as a two-sided ideal rather than a left $A$–module) then $A/J$ is a semisimple algebra.

If our field is algebraically closed, then the Artin-Wedderburn Theorem says that $A/J \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$, and we say that $A$ is *elementary* if $n_i = 1$ for all $i$.

Another fact about the Jacobson radical which I want to use is the following. Recall that if $I$ is a two-sided ideal in some ring then $I^k := \{a_1 a_2 \cdots a_k : a_r \in I\}$.

**Theorem 9.1.** *If $A$ is a finite-dimensional algebra with Jacobson radical $J$ then for all $k$, either $J^k = 0$ or $J^k \supsetneq J^{k+1}$.*

*Proof.* If $J^k = J^{k+1}$ then $J^k = JJ^{k+1} = \mathrm{rad}(A)J^{k+1}$. But recall that for every $A$–module $M$, $\mathrm{rad}(A)M \subset \mathrm{rad}(M)$. So $J^k \subset \mathrm{rad}(J^k)$. By definition of the radical, this is impossible unless $J^k = 0$. $\square$

The series of ideals

$$A \supset J \supset J^2 \supset \cdots$$

must therefore end in 0. It is called the *Loewy series* for $A$. It is not necessarily a composition series. The number

$$\ell := \min\{m : J^m = 0\}$$

is called the *Loewy length* of $A$.

9.1. **The quiver of an algebra.** Now let us associate a quiver to an elementary algebra $A$. Let $A$ be an elementary algebra over an algebraically closed field $k$. Let $\{e_i\}$ be a complete set of orthogonal primitive idempotents in $A$, so that $1 = \sum e_i$. Let $\varepsilon_i = e_i + J \in A/J$. Then the $\varepsilon_i$ are nonzero primitive idempotents in $A/J$, and $A/J = \bigoplus_i k\varepsilon_i$. We define the quiver $Q(A)$ as follows: the vertices of $Q(A)$ are the idempotents $\varepsilon_i$ and the number of arrows from $i$ to $j$ is equal to

$$\dim_k \left(\varepsilon_j (J/J^2)\varepsilon_i\right).$$

Notice that the quiver could have loops! (Example below).

**Definition 9.2.** *$Q(A)$ is called the* quiver *of $A$.*

**Examples 9.3.** (1) Let

$$A = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}$$

then the set of nilpotent elements in $A$ is

$$J := \begin{pmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ k & k & 0 \end{pmatrix}$$

and this happens to be a two-sided ideal. Since the Jacobson radical is the largest ideal consisting entirely of nilpotents, $J$ must be equal to the Jacobson radical. We also have that

$$J^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$$

and so $J/J^2$ may be identified with the two-dimensional vector space

$$\begin{pmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & k & 0 \end{pmatrix}$$

The idempotents $\varepsilon_i$ are $\varepsilon_1 = \mathrm{diag}(1,0,0)$, $\varepsilon_2 = \mathrm{diag}(0,1,0)$, $\varepsilon_3 = \mathrm{diag}(0,0,1)$ modulo $J$. A quick calculation shows that

$$\dim(\varepsilon_j(J/J^2)\varepsilon_i) = \begin{cases} 1 & j = 3, i = 2 \text{ or } j = 2, i = 1. \\ 0 & \text{otherwise.} \end{cases}$$

The quiver $Q(A)$ is

$$1 \longrightarrow 2 \longrightarrow 3.$$

(2) Let

$$A = \{ \begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & 0 & b \end{pmatrix} : a, b, c, d \in k\} \subset M_3(k).$$

Then the Jacobson radical is

$$J = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

and this time $J^2 = 0$. There are two idempotents, $e_1 = \mathrm{diag}(1,0,0)$, $e_2 = \mathrm{diag}(0,1,1)$ and this time we get

$$Q(A) = \begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2. \end{tikzcd}$$

(3) Let $A = \bigwedge \mathbb{C}^3$, the exterior algebra of a three-dimensional vector space. Then

$$A = \mathbb{C}1 \oplus \wedge^1\mathbb{C} \oplus \wedge^2\mathbb{C} \oplus \wedge^3 \mathbb{C}.$$

The product is given by $\wedge$. Let $\{x, y, z\}$ be a basis for $\mathbb{C}^3$.
If $a \in \mathbb{C}^3$ then $a^2 = a \wedge a = 0$, and therefore every element of $\bigoplus_{i>0} \wedge^i\mathbb{C}$ is nilpotent. Since this forms a two-sided ideal and it consists precisely of the nilpotent elements, it must be the Jacobson radical $J$. Then $J^2 = \wedge^2\mathbb{C} \oplus \wedge^3\mathbb{C}$ and $J/J^2$ is spanned by the images of $x, y$ and $z$. We have $A/J = \mathbb{C}$ and

there is one idempotent $\varepsilon_1$. The space $\varepsilon_1(J/J^2)\varepsilon_1$ is three-dimensional. Thus, the quiver $Q(A)$ has one vertex and three loops. This illustrates that $Q(A)$ may have oriented cycles.

**Theorem 9.4.** *Let $A$ be a finite-dimensional elementary algebra over an algebraically closed field $k$. Then there is a surjective algebra map*

$$kQ(A) \to A$$

*where $Q(A)$ denotes the quiver of $A$. The kernel $K$ of this surjection satisfies*

$$R^s \subset K \subset R^2$$

*for some $s \geq 2$, where $R$ denotes the arrow ideal of $kQ(A)$.*

*Proof.* Today we will prove everything except the surjectivity. Let $J$ be the Jacobson radical of $A$ and let $e_1, \ldots, e_n$ be a set of primitive orthogonal idempotents such that $\sum_i e_i = 1$. Let $\varepsilon_i$ be the image of $e_i$ in $A/J$, $\varepsilon_i = e_i + J$. Then $A/J = \oplus k\varepsilon_i$. Now, for each $i, j$, let $\{\overline{(y_{ij})}_s\}$ be a basis of $\varepsilon_j(J/J^2)\varepsilon_i$. There is a map

$$\begin{aligned} e_j J e_i &\to \varepsilon_j(J/J^2)\varepsilon_i \\ x &\mapsto x + J^2. \end{aligned}$$

Let $\{(y_{ij})_s\} \subset e_j J e_i$ be a set of elements such that $(y_{ij})_s + J^2 = \overline{(y_{ij})}_s$ for each $i, j, s$. Then the set of vertices of $Q(A)$ may be identified with $\{\varepsilon_i\}$ and the set of arrows with $\{\overline{(y_{ij})}_s\}$. We define a linear map of vector spaces as follows.

$$\begin{aligned} \varepsilon_i &\mapsto e_i \\ \overline{(y_{ij})}_s &\mapsto (y_{ij})_s. \end{aligned}$$

The domain of this map is the space spanned by the paths of length $\leq 1$ in $kQ(A)$ and its range is $A$. Here, we are abusing notation slightly and writing $\varepsilon_i$ for the trivial path at the vertex $\varepsilon_i$, when we should perhaps have written $e_{\varepsilon_i}$. In order to get the map $f$, we simply extend our map to paths of arbitrary length in the obvious way; a path in $Q(A)$ is a sequence of arrows, and we map this sequence to the product of the image of each of the arrows in the specified order. We now have a linear map defined on the natural basis of $kQ(A)$, and we should check that this is an algebra map. But this just amounts to checking that the relations in $kQ(A)$ are preserved, which is quite obvious by construction and the choice of the $(y_{ij})_s$.

So we have a map of algebras $f : kQ(A) \to A$. We don't yet know that this is surjective, but we can prove the statement about the kernel. We want to show that the kernel is a subset of $R^2$.

Let $x \in \ker(f)$. We may write

$$x = \sum \alpha_i \varepsilon_i + \sum \beta_{ijs} \overline{(y_{ij})}_s + \gamma$$

where $\alpha_i$ and $\beta_{ijs} \in k$, and $\beta \in R^2$. Then

$$f(x) = \sum \alpha_i e_i + \sum \beta_{ijs} (y_{ij})_s + f(\gamma) = 0 \in A.$$

If we look at the image of $f(x)$ in $A/J$, we get $f(x) + J = \sum \alpha_i \varepsilon_i = 0$. But $\varepsilon_i$ are a basis for $A/J$, so $\alpha_i = 0$ for all $i$. Therefore

$$f(x) = \sum \beta_{ijs} (y_{ij})_s + f(\gamma) = 0 \in A.$$

Now if we look at $f(x) + J^2$, we see that $f(x) + J^2 = \sum \beta_{ijs} \overline{(y_{ij})}_s = 0$ and hence $\beta_{ijs} = 0$ for all $i, j, s$. So $x = \gamma \in R^2$, as required.

Finally, since $A$ is finite-dimensional, $J^\ell = 0$ where $\ell$ is the Loewy length of $A$. We have $f(R^\ell) \subset J^\ell$ since $f$ is an algebra map. Thus, $R^\ell \subset \ker(f)$.

This proves everything except for the surjectivity of $f$, which we will prove in the next lecture. $\square$

## 10. Lecture 10

Recap: $A$ is an elementary algebra over an algebraically closed $k$. $1 = \sum e_i$ is a decomposition of $1 \in A$ into pairwise orthogonal primitive idempotents. We define $\varepsilon_i = e_i + J \in A/J$. Note that the $\varepsilon_i$ form a basis for $A/J$, because they are a linearly independent set in $A/J$, and our assumption that $A$ is elementary implies that $\dim(A/J)$ is equal to the number of $e_i$ (see the proof of the Artin-Wedderburn Theorem above). The quiver $Q(A)$ has vertices $\varepsilon_i$ and the number of edges $i \to j$ is $\dim(\varepsilon_j (J/J^2) \varepsilon_i)$ for all $i, j$. We defined a map $f : kQ(A) \to A$ by

$$\varepsilon_i \mapsto e_i$$

$$\overline{(y_{ij})}_s \mapsto (y_{ij})_s$$

where $\{\overline{(y_{ij})}_s\}$ is a $k$–basis of $\varepsilon_j (J/J^2) \varepsilon_i$ and $\{(y_{ij})_s\} \subset e_j J e_i$ are chosen so that $(y_{ij})_s + J^2 = \overline{(y_{ij})}_s$. We proved that $f$ extends to a well-defined algebra map and that the kernel $K$ satisfies $R^s \subset K \subset R^2$ for some $s \geq 2$ where $R$ is the arrow ideal of $Q(A)$.

Now we will show that $f$ is surjective. The image of $f$ is the subalgebra of $A$ generated by the $e_i$ and the $(y_{ij})_s$. The elements $\{(y_{ij})_s + J^2\}$ for all $i, j, s$ form a basis for the space $J/J^2 = \bigoplus_{i,j} \varepsilon_j (J/J^2) \varepsilon_i$. It therefore suffices to prove the following lemma, taken from [ARS97].

**Lemma 10.1.** [ARS97, Theorem 1.9(a)] *Let $A$ be a finite-dimensional elementary algebra. Let $e_1, \dots, e_n$ be a set of primitive orthogonal idempotents such that $\sum e_i = 1$. Let $J$ be the Jacobson radical of $A$ and let $\{r_1, \dots, r_m\} \subset J$ be such that $\{r_1 + J^2, \dots, r_m + J^2\}$ is a generating set of the $A/J$–module $J/J^2$. Then $\{e_1, \dots, e_n, r_1, \dots, r_m\}$ generates $A$ as an algebra.*

*Proof.* The proof is by induction on the Loewy length $\ell$ of $A$. If $\ell = 1$ then $J = 0$. In this case, $A \cong k \times k \times \cdots \times k$ and the $e_i$ are the elements $(0, 0, \dots, 1, \dots, 0)$ and these generate $A$.

If $\ell = 2$ then $J^2 = 0$ and the hypothesis says that $\{r_1, \ldots, r_m\}$ generate $J$ as an $A/J$–module. If $a \in A$ then $a + J = \sum \lambda_i e_i + J$ for some $\lambda_i \in k$. Therefore, $a - \sum \lambda_i e_i \in$ the $k$–span of the $e_i r_j$, which shows that the algebra $A$ is generated by $\{e_i\} \cup \{r_j\}$ in this case too.

Now we assume $\ell \geq 2$ and do the inductive step. Suppose we have shown it for $\ell \leq m$ and we want to show it for $\ell = m + 1$. Let $B \subset A$ be the subalgebra generated by the $e_i$ and the $r_j$. We wish to show that $B = A$. To do this, we consider $A/J^m$. This is an algebra. Because each maximal left ideal of $A$ contains $J^m$, the Jacobson radical of $A/J^m$ is $J/J^m$, and $(J/J^m)^m = 0$. Also, $A/J^m$ is elementary because $(A/J^m)/(J/J^m) = A/J$. Furthermore, the $e_i + J^m$ are primitive orthogonal idempotents in $A/J^m$. (I left this last fact as an exercise in the lecture. It is true because the isomorphism $(A/J^m)/(J/J^m) = A/J$ shows that the maximum possible number of orthogonal idempotents in a decomposition $1 = \sum f_i$ is the same for $A/J^m$ as for $A$.) So, in summary, we may apply the inductive hypothesis to $A/J^m$.

We conclude that the natural map

$$B/B \cap J^m \to A/J^m$$

is an isomorphism.

Now let $x \in A$. Then there exists $y \in B$ with $x - y \in J^m$. So $x - y = \sum \alpha_i \beta_i$ where $\alpha_i \in J$ and $\beta_i \in J^{m-1}$. Consider $\alpha_i$. There exists some $a_i \in B$ with $\alpha_i - a_i \in J^m$. So $\alpha_i = a_i + a_i'$ where $a_i' \in J^m$ and $a_i \in B \cap J$ ($a_i$ has to be in $J$ because $a_i = \alpha_i - a_i' \in J$). Similarly, $\beta_i = b_i + b_i'$ where $b_i \in B \cap J^{m-1}$ and $b_i' \in J^m$. Thus,

$$x - y = \sum (a_i + a_i')(b_i + b_i')$$

with $a_i b_i' \in J^{m+1}$, $a_i' b_i' \in J^{2m}$ and $a_i' b_i \in J^{2m-1}$. But these powers of $J$ are all zero, because $J^{m+1} = 0$ and our assumption that $m \geq 2$ shows that $2m - 1 \geq m + 1$. Therefore,

$$x - y = \sum a_i b_i \in B$$

and therefore $x \in B$ as desired. $\square$

Using the very clever argument above, copied verbatim from [ARS97], we have finished the proof of the following Theorem.

**Theorem 10.2.** *Let $A$ be a finite-dimensional elementary algebra over an algebraically closed field $k$. Then there is a surjective algebra map*

$$kQ(A) \to A$$

*where $Q(A)$ denotes the quiver of $A$. The kernel $K$ of this surjection satisfies*

$$R^s \subset K \subset R^2$$

*for some $s \geq 2$, where $R$ denotes the arrow ideal of $kQ(A)$.*

Now we want to say a bit more about the ideal $K$.

**Definition 10.3.** *A* quiver with relations *is a pair $(Q, \rho)$ where $Q$ is a quiver and $\rho$ is a* finite *subset of nonzero elements of $kQ$ such that each element of $\rho$ is of the form $e_i x e_j$ where $i, j \in Q_0$ and $x$ is a sum of paths of length $\geq 2$.*

We usually also insist that $R^s \subset \langle \rho \rangle$ where $R^s$ denotes the arrow ideal of $kQ$. In this case, $kQ/\langle \rho \rangle$ is a finite-dimensional algebra, because it is a vector subspace of $kQ/R^s$, which is spanned by the paths in $Q$ of length $< s$.

We have almost shown that every finite-dimensional algebra is given by a quiver and relations, but we just need to make sure that we can avoid any illegal relations.

**Theorem 10.4.** *Every elementary algebra $A$ is isomorphic to $kQ/\langle \rho \rangle$ where $(Q, \rho)$ is a quiver with relations.*

*Proof.* Let $f : kQ(A) \to A$ be the surjection we constructed above and let $K = \ker(f)$. We need to show that $K$ is a finitely-generated ideal of $kQ(A)$. Let $\{b_i + R^s\}_{i=1}^n$ be a basis of $K/R^s$. Then $K = \sum kQ(A) b_i kQ(A) + R^s$. But $R^s$ is generated by all paths of length $s$, so $K$ is generated by the finite set of all $b_i$ together with all paths of length $s$. Therefore, $K$ is generated by some finite collection $a_1, a_2, \ldots, a_N$. But if $K$ is generated by these elements, then it is also generated by $\{e_r a_i e_s : 1 \leq i \leq N\}$ where $r, s$ range throughout all vertices of $Q(A)$. Each $e_r a_i e_s$ is of the desired form, so $\rho = \{e_r a_i e_s\}$ are the desired relations. $\square$

A next obvious question is: how unique is the pair $(Q, \rho)$ associated to a given $A$? The surprising answer is that $Q$ is unique. But $\rho$ is not unique. Here is an example.

**Example 10.5.** Let $Q$ be the quiver

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xleftarrow{\gamma} \bullet \xleftarrow{\delta} \bullet \; .$$

Then clearly $\langle \beta\alpha \rangle \neq \langle \gamma\delta \rangle$, but $kQ/\langle \beta\alpha \rangle \cong kQ/\langle \gamma\delta \rangle$.

In order to show that the quiver associated to a given $A$ is unique, we require the following lemma.

**Lemma 10.6.** *If $Q$ is any quiver and $R$ is the arrow ideal of $kQ$ then the quiver of $kQ/R^2$ is $Q$.*

*Proof.* Let $A = kQ/R^2$. We claim that the Jacobson radical of $A$ is $R/R^2$. Since $A$ is a finite-dimensional algebra, the Jacobson radical is the largest ideal consisting of nilpotent elements. Certainly $R/R^2$ is such an ideal. On the other hand, if $x \in A$ is nilpotent then $x = \sum \lambda_i e_i + y$ where $\lambda_i \in k, y \in R$. Then $x^n = \sum_i \lambda_i^n e_i$ modulo $R$, so if $x$ is nilpotent then $\lambda_i = 0$ for all $i$. Thus, $R/R^2$ is precisely the set of nilpotent elements of $A$, so it is the Jacobson radical.

The quiver $Q(A)$ has vertices $\varepsilon_i = e_i + J$ where $e_i$ are a complete set of orthogonal primitive idempotents in $A$. We have $1 = \sum \overline{e_i}$ in $A$ (where we write $\overline{e_i}$ for the image of $e_i \in kQ$ in $A$), and the $\overline{e_i}$ are orthogonal idempotents, but we need to check they are primitive. This can be done using Fitting's Lemma: $\mathrm{End}(A\overline{e_i})$

is isomorphic to $\overline{e_i}A\overline{e_i}$, a vector space spanned by the arrows in $Q$ joining $i$ to $i$. This need not be a one-dimensional space, because we allow $Q$ to have loops. However, every element is of the form $\alpha\overline{e_i} + n$ where $\alpha \in k$ and $n \in R$ is nilpotent. If $\alpha = 0$ then such an element is nilpotent. If $\alpha \neq 0$ then such an element is a unit, because we have the formula for the inverse of such an element in any ring, $(1-n)^{-1} = 1+n+n^2+\cdots$. Thus, every element of $\mathrm{End}(A\overline{e_i})$ is either a unit or nilpotent, so $A\overline{e_i}$ is an indecomposable module, and $\overline{e_i}$ is a primitive idempotent.

Thus, the $\varepsilon_i$ are in bijection with the vertices of $Q$, and so the quiver of $A$ has the same vertices as $Q$. The arrows in the quiver from $i$ to $j$ are in bijection with a basis of the space $\varepsilon_j(R/R^2)\varepsilon_i$, but this is the same as the number of arrows from $i$ to $j$ in $Q$, as required. □

Now we prove the uniqueness of the quiver of an algebra.

**Theorem 10.7.** *Let $(Q,\rho)$ and $(Q',\rho')$ be quivers with relations, such that $R^s \subset \langle\rho\rangle \subset R^2$ and $(R')^s \subset \langle\rho'\rangle \subset (R')^2$ where $R$ and $R'$ are the arrow ideals of $Q$ and $Q'$ respectively. Suppose the algebras $kQ/\langle\rho\rangle$ and $kQ'/\langle\rho'\rangle$ are isomorphic. Then $Q$ and $Q'$ are isomorphic quivers.*

*Proof.* The Jacobson radical of $kQ/\langle\rho\rangle$ is $R/\langle\rho\rangle$. This can be proved using the same argument as given in Lemma 10.6. Thus, if $A := kQ/\langle\rho\rangle$ then $A/\mathrm{rad}(A)^2$ is isomorphic to both $kQ/R^2$ and $kQ'/(R')^2$, so by Lemma 10.6, the quiver of $A/\mathrm{rad}(A)^2$ is isomorphic to both $Q$ and $Q'$. □

## 11. Lecture 11

11.1. **Dealing with quivers with relations.** It turns out that it is quite easy to work out the simple, indecomposable projective and indecomposable injective modules for a quiver with relations. It is very similar to the case of a path algebra with no relations.

Let $(Q,\rho)$ be a quiver with relations. A *representation* of $(Q,\rho)$ consists of

- A vector space $V_i$ for each $i \in Q_0$.
- A linear map $\phi_a : V_{t(a)} \to V_{h(a)}$ for each $a \in Q_1$, such that if $x \in \rho$ then $\phi_x = 0$.

The condition $\phi_x = 0$ means the obvious thing. For example, if $x$ is the relation $ab + 3cd = 0$ where $a, b, c, d$ are arrows, then $\phi_x$ denotes $\phi_a\phi_b + 3\phi_c\phi_d$.

It is easy to see that representations of $(Q,\rho)$ are the same thing as $kQ/\langle\rho\rangle$–modules.

Let $(Q,\rho)$ be a quiver with relations and suppose $R^s \subset \langle\rho\rangle \subset R^2$ as usual. We now work out the indecomposable projective left modules over the algebra $A := kQ/\langle\rho\rangle$. We have the trivial idempotents $e_i \in kQ$ and their images $\overline{e_i} := e_i + \langle\rho\rangle$ in $kQ/\langle\rho\rangle$. The $\overline{e_i}$ are mutually orthogonal, sum to 1, and remain primitive by the argument that was used in Lemma 10.6 (here we use the fact that $R^s \subset \langle\rho\rangle$ which guarantees that every element of $R$ is nilpotent in $A$). Thus, the principal indecomposable modules are $A\overline{e_i}$ for $i \in Q_0$. Just as in the no-relations case, the radical of $A\overline{e_i}$ is $\oplus_{a\in Q_1} A\overline{a}\overline{e_i}$ where $\overline{a}$ denotes the image of $a$ in the quotient $kQ/\langle\rho\rangle$. The head of $A\overline{e_i}$ is then the representation of $(Q,\rho)$ given by $k$ at the vertex $i$ and 0 at all the other vertices.

**Example 11.1.** Let $Q$ be the following quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha"] & \bullet \arrow[l, bend left, "\delta"] \arrow[r, "\beta"] & \bullet \arrow[loop right, "\gamma"] \end{tikzcd}$$

with the relations $\rho = \langle \gamma\beta, \gamma^2, \delta\alpha\rangle$. A basis of $A = kQ/\langle\rho\rangle$ is given by $e_0, e_1, e_2, \alpha, \beta, \gamma, \delta, \beta\alpha, \beta\alpha\delta, \alpha\delta$ (we drop the bars for the images of elements of $kQ$ in $A$), so this is a 10–dimensional algebra. If we label the vertices $0, 1, 2$ from left to right, then it is easy to compute a basis of each indecomposable projective $Ae_i$. A basis for $Ae_2$ is $e_2, \gamma$, a basis for $Ae_1$ is $e_1, \delta, \beta, \alpha\delta, \beta\alpha\delta$ and a basis for $Ae_0$ is $e_0, \alpha, \beta\alpha$.

11.2. **Hereditaryness.** Now finally we can prove that hereditary elementary algebras are precisely the path algebras of cycle-free quivers.

**Theorem 11.2.** *If $A$ is a finite-dimensional hereditary algebra over an algebraically closed field $k$ then there is a quiver $Q$ with $A \cong kQ$.*

*Proof.* Let $Q := Q(A)$ be the quiver of $A$. We show that $Q$ has no cycles. To see this, suppose that $\varepsilon_j(J/J^2)\varepsilon_i \neq 0$ where $J$ denotes the Jacobson radical of $A$, and $\varepsilon_i = e_i + J$, where $\{e_i\}$ is a complete set of orthogonal primitive idempotents in $A$. If $\varepsilon_j(J/J^2)\varepsilon_i \neq 0$ then $e_jJe_i \neq 0$ and so if $x \in e_jJe_i$ is nonzero, then $f : a \mapsto ax$ is a nonzero $A$–map from $Ae_j$ to $Ae_i$. The map $f$ is *not* an isomorphism, or else $e_i$ would be in the image and we would have $e_i \in J$. But $\mathrm{Im}(f) \subset Ae_i$ is a submodule of a projective. Since $A$ is hereditary, $\mathrm{Im}(f)$ is projective, and so the sequence

$$0 \to \ker(f) \to Ae_j \to \mathrm{Im}(f) \to 0$$

splits. But $Ae_j$ is indecomposable, and therefore $f : Ae_j \to \mathrm{Im}(f)$ must be an isomorphism. So $f$ is injective, but not an isomorphism.

Now if $Q$ contains a loop starting and ending at $i$, we get a sequence of such $f$'s. Their composition is an injective map $Ae_i \to Ae_i$ which is not an isomorphism. Since everything is finite-dimensional, this violates linear algebra.

Thus, $Q$ has no loops and so $kQ$ is finite-dimensional (and hereditary). ∎

## 12. Lecture 12

Let us finish the proof that every hereditary elementary algebra is isomorphic to the path algebra of a quiver. So far, we have shown that if $A$ is an elementary hereditary algebra then the quiver $Q = Q(A)$ of $A$ has no loops. We know that $A$ is isomorphic to $kQ/\langle\rho\rangle$ for some relations $\rho$, and we want to show that $\rho = 0$. This is a consequence of the following theorem from [ARS97].

**Theorem 12.1.** [ARS97, Lemma 1.11] *Let $A$ be a finite-dimensional hereditary algebra, let $J = \mathrm{rad}(A)$ be its Jacobson radical, and suppose that $I$ is a two-sided ideal of $A$ with $I \subset J^2$. Then either $I = 0$ or $A/I$ is not hereditary.*

*Proof.* Assume $A/I$ is hereditary. We have the ideal $IJ = \{\sum a_i b_i : a_i \in I, b_i \in J\}$. This two-sided ideal is contained within both $I$ and $J$. Consider the short exact sequence of left $A/I$–modules:

$$0 \to I/IJ \to J/IJ \to J/I \to 0.$$

The last term $J/I$ is projective because $A/I$ is hereditary. Denote by $f : J/IJ \to J/I$ the natural map (which appears in the above sequence). This map splits and so $J/IJ$ is the sum of the submodules $\ker(f) = I/IJ$ and another submodule $T$ which is isomorphic to $J/I$. But $I \subset J^2$ implies that $I/IJ \subset \mathrm{rad}(A/I)(J/IJ) \subset \mathrm{rad}(J/IJ)$. Now, if $T \neq J/IJ$ then $T$ is contained in some maximal submodule and this is impossible. So $T = J/IJ$ and $f$ is an isomorphism. Thus, $I/IJ = 0$ and so $I = IJ$. But $IJ = I\mathrm{rad}(A) \subset \mathrm{rad}(I)$, where rad here denotes the radical in the category of *right* $A$–modules, and this is impossible unless $I = 0$. □

Theorem 11.2 now follows immediately from the above.

Note: in the above proof, we showed in class that $J/IJ$ was projective as well. But actually, we didn't need this, as one of you pointed out.

12.1. **Morita Theory.** We have classified the possible elementary algebras via quivers and relations. Now we want to see how arbitrary algebras fit in. To do this, we use Morita Theory.

**Definition 12.2.** *If $A$ and $B$ are algebras, $A$ and $B$ are said to be* Morita equivalent *if $A-\mathrm{mod}$ and $B-\mathrm{mod}$ are equivalent categories.*

An example of Mortia equivalent algebras which are not isomorphic are $k$ and $M_2(k)$ where $k$ is a field. These two algebras are not isomorphic because $M_2(k)$ is not commutative, but the module category of $k$ is a semisimple category with one simple object $k$ and $\mathrm{Hom}(k, k) = k$, while the module category of $M_2(k)$ is a semisimple category with one simple object $k^2$, and $\mathrm{Hom}_{M_2(k)}(k^2, k^2) = k$.

We will show that every finite-dimensional algebra over $\mathbb{C}$ is Morita equivalent to an elementary algebra.

**Definition 12.3.** *A module $M \in A-\mathrm{mod}$ is called a* generator *if every $N \in A-\mathrm{mod}$ is a quotient of a direct sum of copies of $M$.*

For example, $A$ itself is always a generator.

**Theorem 12.4.** *Let $A$ be a finite-dimensional algebra over an algebraically closed field $k$. Let $P$ be a projective generator in $A-\mathrm{mod}$. Then there is an equivalence of categories*

$$F : A-\mathrm{mod} \to \mathrm{mod}-S$$

*where* $S = \mathrm{End}_A(P)$. *Then functor* $F$ *is* $\mathrm{Hom}_A(P, -)$ *and its quasiinverse is* $G : \mathrm{mod} - S \to A - \mathrm{mod}$ *given by* $G(N) = N \otimes_S P$.

*Proof.* In the theorem, the functors $F$ and $G$ make sense, because $P$ is regarded as both a left $A$–module and a left $S$–module, while $N \otimes_S P$ is a left $A$–module with the action of $r \in A$ given by $r(n \otimes p) = n \otimes rp$.

Both $F$ and $G$ are functors and they preserve finite direct sums. Also, both $F$ and $G$ are right exact, and therefore so are $FG$ and $GF$.

We have $FG(S) = F(P) = S$, so $FG(\bigoplus_{i=1}^n S) = \bigoplus_{i=1}^n FG(S)$. Now suppose $M$ is any right $S$–module. There is a natural map $M \to FG(M)$ given by $m \mapsto (p \mapsto m \otimes p) \in \mathrm{Hom}_A(P, M \otimes P)$. We wish to show that this map is an isomorphism. There is an exact sequence

$$\bigoplus_{a \in A} S \to \bigoplus_{b \in B} S \to M \to 0$$

where $A$ and $B$ are finite sets. Using naturality of the map $U \to FG(U)$ for any $U$, we get the following commutative diagram.

$$\begin{tikzcd}
\bigoplus_{a\in A} S \arrow[r] \arrow[d] & \bigoplus_{b\in B} S \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\bigoplus_{a\in A} FG(S) \arrow[r] & \bigoplus_{b\in B} FG(S) \arrow[r] & FG(M) \arrow[r] & 0
\end{tikzcd}$$

The lower row is exact because $FG$ is a right exact functor. The first two vertical maps are isomorphisms, and the five-lemma then implies that the map $M \to FG(M)$ is an isomorphism as well. Since $M$ was arbitrary, this proves that $FG$ is naturally isomorphic to the identity functor.

To do the same for $GF$, observe that $GF(P)$ is isomorphic to $P$. If $M$ is an $R$–module then there is a natural map $GF(M) \to M$. Furthermore, there is an exact sequence

$$\bigoplus_{a \in A} P \to \bigoplus_{b \in B} P \to M \to 0$$

where $A$ and $B$ are finite sets. This is because $P$ is a generator. As before, we get a diagram with exact rows

$$\begin{tikzcd}
\bigoplus_{a\in A} P \arrow[r] & \bigoplus_{b\in B} P \arrow[r] & M \arrow[r] & 0 \\
\bigoplus_{a\in A} GF(P) \arrow[r] \arrow[u] & \bigoplus_{b\in B} GF(P) \arrow[r] \arrow[u] & GF(M) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and we may apply the five lemma as before to get that $GF(M) \to M$ is an isomorphism. We conclude that $GF$ is naturally isomorphic to the identity functor, and thus $F$ and $G$ are quasiinverse equivalences of categories. $\square$

**Remark 12.5.** The above theorem is part of *Morita's Theorem* which says that any equivalence between the module categories of a pair of rings $R$ and $S$ arises from a projective generator in a similar way. The definition of a generator is a little different if you are not dealing with finitely-generated modules. See [MR01] for details.

The above proof was taken almost word-for-word from [Rot09].

12.2. **Application.** Let $A$ be a finite-dimensional algebra. Then $A = P_1^{a_1} \oplus \cdots \oplus P_r^{a_r}$ where the $P_i$ are the principal indecomposable $A$–modules. Take $P := P_1 \oplus \cdots \oplus P_r$. Then $P$ is a projective generator. Let $S = \mathrm{End}_A(P)$. Then $A - \mathrm{mod}$ is equivalent to $S^{op} - \mathrm{mod}$. Now, $S = F(P) = F(P_1) \oplus \cdots \oplus F(P_r)$, and $F$ is an equivalence of categories, which implies that the $F(P_i)$ are pairwise nonisomorphic indecomposable projectives. So the multiplicity of each indecomposable summand of $S$ as a right $S$–module is one. Therefore, the same is true of the left $S^{op}$–module $S^{op}$. Now the proof of the Artin-Wedderburn Theorem shows that this implies that $S^{op}/J$ is a product of copies of the base field $k$, where $J$ is the Jacobson radical of $S^{op}$. Thus, we have shown the following.

**Theorem 12.6.** *If $A$ is a finite-dimensional algebra over an algebraically closed field $k$, then $A$ is Morita equivalent to an elementary algebra.*

**Corollary 12.7.** *Every finite-dimensional $\mathbb{C}$–algebra is Morita equivalent to $\mathbb{C}Q/\langle\rho\rangle$ where $(Q, \rho)$ is a quiver with relations.*

This completes this part of the course.

**Exercise 12.8.** Here is an interesting exercise. Let $A$ be an algebra and let $Z(A) = \{x \in A : xy = yx \text{ for all } y \in A\}$.

(1) Show that the set $\mathrm{Nat}(\mathrm{id}, \mathrm{id})$ of natural transformations $\mathrm{id} \to \mathrm{id}$, where $\mathrm{id} : A - \mathrm{mod} \to A - \mathrm{mod}$ is the identity functor, forms an algebra and that the algebra $\mathrm{Nat}(\mathrm{id}, \mathrm{id})$ is isomorphic to $Z(A)$.
(2) If $A$ and $B$ are commutative algebras and $A$ and $B$ are Morita equivalent, show that $A \cong B$.

**Exercises 12.9.** Here are some exercises on the material covered above.

(1) Find the simple $\mathbb{C}Q$–modules and their projective covers, where $Q$ is the following quiver

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[r] & \bullet \\
 & \bullet \arrow[u] &
\end{tikzcd}$$

(2) Let $k$ be an algebraically closed field. Express the algebra $T_n$ as a quiver and relations where

$$T_n = \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ k & k & 0 & \cdots & 0 \\ k & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ k & 0 & 0 & \cdots & k \end{pmatrix} \subset M_n(k).$$

Express $T_n^{op}$ as a subalgebra of $M_n(k)$ and show that $T_n^{op} \ncong T_n$.

(3) Let $m, n \geq 2$. Show that the algebras $\mathbb{C}[x, y]/(x^n, y^m)$ and $\bigwedge \mathbb{C}^2$ have the same quiver but are not isomorphic.

(4) (a) Show that every three-dimensional $\mathbb{C}$–algebra is elementary.

(b) Write down a complete list of all three-dimensional $\mathbb{C}$–algebras up to isomorphism.

(5) Let $A = \mathbb{C}Q/\langle \rho \rangle$ where $Q$ is the quiver

$$\begin{tikzcd}
 & \bullet \arrow[dr, "\gamma"] & \\
\bullet \arrow[ur, "\alpha"] & & \bullet \arrow[ll, "\beta"']
\end{tikzcd}$$

and $\rho = \alpha\beta\gamma$.

(a) Write down a basis for $A$.

(b) Find the simple $A$–modules and their projective covers.

(c) Write down a composition series for each indecomposable projective.

(d) Show that $A$ is not hereditary.

(e) Calculate the global dimension of $A$.

(6) Find a quiver and relations $(Q, \rho)$ such that the algebra $A$ is Morita equivalent to $kQ/\langle \rho \rangle$, where

$$A = \begin{pmatrix} k & 0 & k \\ 0 & k & 0 \\ k & 0 & k \end{pmatrix}$$

Now we turn to the next part of the course. Remember that we want to understand the module categories of algebras. This means that we want to calculate all the modules and the morphisms between them. The Krull-Schmidt Theorem says that every module is a direct sum of indecomposable modules. If we understand the maps between the indecomposables, then we can understand the maps between direct sums of indecomposables, because maps between direct sums can be regarded as matrices whose entries are the maps between the individual summands, if that makes sense. There are some algebras for which there are only finitely many indecomposables.

**Definition 13.1.** *An algebra $A$ has* finite type *if it has finitely many isomorphism classes of indecomposable modules.*

For algebras of finite type, we can hope to completely describe the module category in terms of finitely much information.

**Example 13.2.** Let $Q$ be the quiver $\bullet \longrightarrow \bullet$ . Some indecomposable $kQ$–modules are $S_0 := k \to 0$, $S_1 := 0 \to k$ and $P_0 := k \to k$ where the map is the identity. Note that $P_0$ is the projective cover of $S_0$. It is also the injective envelope of $S_1$. We claim that this is a complete set of indecomposable $kQ$–modules, so $kQ$ has finite type. To see this, take a general module of the form $M = \; k^m \xrightarrow{A} k^n$ . Consider the following commutative diagram with exact columns.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\ker(A) \arrow[r] \arrow[d] & 0 \arrow[d] \\
k^m \arrow[r, "A"] \arrow[d, "A"] & k^n \arrow[d, "id"] \\
k^n \arrow[r, "id"] \arrow[d] & k^n \arrow[d] \\
0 & 0
\end{tikzcd}$$

The diagram may be regarded as a short exact sequence

$$0 \longrightarrow S_0^{\oplus \dim \ker(A)} \longrightarrow M \longrightarrow P_0^{\oplus n} \longrightarrow 0$$

in the category of representations of $Q$. Sine $P_0$ is projective, so is $P_0^n$, and the sequence splits. Therefore, if $M$ is indecomposable, we must have $\ker(A) = 0$. A similar argument with $\mathrm{cok}(A)$ and using the injectivity of $P_0$ shows that $\mathrm{cok}(A) = 0$ as well. Therefore, if $A \neq 0$ then $A$ is an isomorphism and $m = n$. So $M$ is of the form $k^n \xrightarrow{A} k^n$. But this is isomorphic to $k^n \xrightarrow{id} k^n = P_0^{\oplus n}$ because of the following square.

$$\begin{tikzcd}
k^n \arrow[r, "A"] \arrow[d, "id"] & k^n \arrow[d, "A^{-1}"] \\
k^n \arrow[r, "id"] & k^n
\end{tikzcd}$$

Since $M$ is indecomposable, we have $n = 1$ and $M \cong P_0$. There was also the possibility that $A = 0$. But then $M \cong S_0^m \oplus S_1^n$ and so $M \cong S_0$ or $M \cong S_1$.

Note that it is quite straightforward to calculate the Hom–spaces between $S_0$, $S_1$ and $P_0$, eg. $\mathrm{Hom}(P_0, S_0) = k$, $\mathrm{Hom}(P_0, S_1) = 0$, etc. This gives us a complete description of the module category of $kQ$.

**Example 13.3.** Now consider the following quiver.

$$Q = \bullet \rightrightarrows \bullet$$

It is an exercise to check that the modules

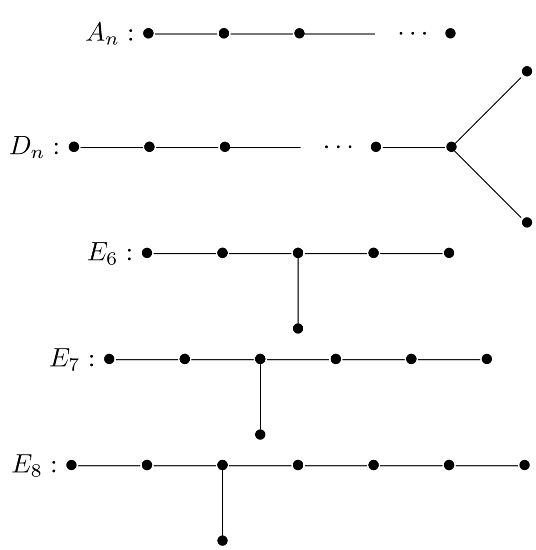
$$M_\lambda := k \underset{1}{\overset{\lambda}{\rightrightarrows}} k$$

for $\lambda \in k$ are indecomposable and $M_\lambda \not\cong M_\mu$ if $\lambda \neq \mu$. So $kQ$ has infinitely many indecomposables. In fact, it turns out that $kQ$ is "tame" and that it is possible to determine its module category completely. We may touch on this later.

13.1. **Gabriel's Theorem.**

**Theorem 13.4** (Gabriel)**.** *If $Q$ is a quiver without oriented cycles then $\mathbb{C}Q$ is of finite type if and only if the underlying graph of $Q$ is an ADE graph.*

By definition, the ADE graphs are the following.



The ADE graphs appear all over mathematics. Perhaps there is some deep reason for this, but I don't understand why. To me, it's just one of those mysteries like Stonehenge or whatever.

The theorem as stated is actually only part of Gabriel's Theorem. We will give a proof due to Gabriel, Tits and Ringel. All the details of the proof are in [CB], and we will follow this very closely. Another popular proof of the theorem is purely algebraic and is due to Gelfand and Ponomarëv.

To prove Gabriel's Theorem, we will show:

(1) Every quiver of finite type is ADE.

(2) Every ADE quiver is of finite type.

The key idea is to use the *Tits form*.

**Definition 13.5.** *Let $Q$ be a quiver, $n = |Q_0|$ the number of vertices of $Q$. The* Tits form *of $Q$ is the quadratic form $q : \mathbb{Q}^n \to \mathbb{Q}$ defined by*

$$q(\alpha) = \sum_{i \in Q_0} \alpha_i^2 - \sum_{a \in Q_1} \alpha_{t(a)} \alpha_{h(a)}.$$

It turns out that the Tits form is positive definite if and only if the underlying graph of $Q$ is ADE (notice that $q$ depends only on the underlying graph and not on the orientation). This is the main fact we will use in the proof of Gabriel's Theorem. We will show that $X \mapsto (\dim(e_i X))_{i \in Q_0}$ is an injection from the set of isomorphism classes of indecomposable representations of $Q$ to the set $\{\alpha \in \mathbb{Z}^n_{\geq 0} : q(\alpha) = 1\}$ (actually it is a bijection) and that the latter set is finite.

In the proof we need to use algebraic geometry, so let us review some of this. A good reference is [Sha94].

An *algebraic set* $V \subset \mathbb{C}^n$ is the set of common zeroes of some polynomials $f_1, \ldots, f_N \in \mathbb{C}[x_1, \ldots, x_n]$. The *coordinate ring* of $V$ is

$$\mathbb{C}[V] := \mathbb{C}[x_1, \ldots, x_n]/\{f : f(x) = 0 \text{ for all } x \in V\}.$$

It can also be described as the ring of functions $f : V \to \mathbb{C}$ such that $f$ is the restriction to $V$ of a polynomial function on $\mathbb{C}^n$.

The *Zariski topology* on $\mathbb{C}^n$ is the topology whose closed sets are the algebraic sets. We will work with *locally closed sets*, that is, those sets of the form $U \cap C$ where $U$ is open in the Zariski topology and $C$ is closed. When I say "variety", I mean a locally closed set in $\mathbb{C}^n$.

A topological space is called *irreducible* if any two nonempty open sets have nonempty intersection. When we say a locally closed $V \subset \mathbb{C}^n$ is irreducible, we mean that $V$ is irreducible when regarded as a topological space with the subspace topology inherited from the Zariski topology on $\mathbb{C}^n$.

The *dimension* $\dim(V)$ of a locally closed $V \subset \mathbb{C}^n$ is the largest $n$ such that there exists a strictly increasing chain of irreducible closed subsets

$$\varnothing \subset C_0 \subset C_1 \subset \cdots \subset C_n$$

of $V$. For every such $V$, there is a nonempty open subset $U \subset V$ such that $U$ is a complex manifold, and $\dim(V)$ is the dimension of this manifold. So if you haven't seen this notion before, the main point is that it behaves as you would expect. For example if $V = \{xy = 0\}$, the union of the two coordinate axes in $\mathbb{C}^2$, then the dimension of $V$ is 1.

If $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ are locally closed sets, then a function $f : V \to W$ is called *regular* if $f = (f_1, \ldots, f_m)$ where each $f_i \in \mathbb{C}(x_1, x_2, \ldots, x_n)$, the field of rational functions on $\mathbb{C}^n$, and all the $f_i$ are well-defined at every point of $V$.

An *algebraic group* is a locally closed set $G \subset \mathbb{C}^N$ such that there are regular maps

$$m : G \times G \to G$$
$$i : G \to G$$

and an element $1 \in G$ such that the group axioms are satisfied, where $m$ is multiplication and $i$ is inversion. An example is $GL_n(\mathbb{C}) \subset \mathbb{C}^{n^2}$. This is an open set because it is given by the non-vanishing of the polynomial function det. The multiplication map is given by polynomial functions. The inversion $G \to G$ sends a matrix $A$ to $\frac{1}{\det(A)}\mathrm{adj}(A)$. Each entry of this matrix is a rational function which is well-defined on the whole of $G$, so this is also a regular map.

An *action* of an algebraic group $G$ on a locally closed $V \subset \mathbb{C}^n$ is a regular map $G \times V \to V$ which is a group action.

13.2. **Our situation.** We want to apply the above in the following situation. Let $Q$ be a quiver and $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0} = \mathbb{Z}_{\geq 0}^n$ where $n = |Q_0|$.

**Definition 13.6.** *Define* $\mathrm{Rep}(Q, \alpha)$ *to be the set of representations of* $Q$ *with dimension vector* $\alpha$*, where if* $V$ *is a representation of* $Q$*, the* dimension vector *of* $V$ *is the vector* $\underline{\dim}(V) := (\dim(V_i))_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$.

We may regard $\mathrm{Rep}(Q, \alpha)$ as a copy of $\mathbb{C}^N$ where $N = \sum_{a \in Q_1} \alpha(h(a))\alpha(t(a))$, because a representation of dimension vector $\alpha$ is just a collection of matrices of size $\alpha(h(a)) \times \alpha(t(a))$ for each arrow $a$.

**Definition 13.7.** *We define*

$$GL(\alpha) = \prod_{i \in Q_0} GL(\alpha_i).$$

The algebraic group $GL(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ as follows. If $\phi = (\phi_a)_{a \in Q_1}$ is a representation of dimension vector $\alpha$ and $g = (g_i)_{i \in Q_0} \in GL(\alpha)$ then we define

$$g \cdot \phi = (g_{h(a)}^{-1} \phi_a g_{t(a)})_{a \in Q_1}.$$

The orbits of this action are precisely the isomorphism classes of representations of $Q$ of dimension vector $\alpha$ (often we loosely say "of dimension $\alpha$"). It is worth checking this because we will use it a lot.

## 14. Lecture 14

Recall our situation: $G$ is an algebraic group acting on an irreducible variety $X$ (in our example, $X = \mathbb{C}^N$). Assume $G$ is also irreducible.

**Theorem 14.1.** *The following are facts:*

1. *For all $x \in X$, the orbit $G \cdot x$ is locally closed in $X$.*
2. $\dim(G \cdot x) = \dim(G) - \dim(\mathrm{stab}_G(x))$.

*Proof.* (Modulo some algebraic geometry).

Consider the action map $G \to G \cdot x$. This map is regular and its image is $G \cdot x$, so $G \cdot x$ is an irreducible topological space. Therefore, the closure $\overline{G \cdot x}$ of $G \cdot x$ in $X$ is also irreducible. Now apply:

*Chevalley's Theorem*: *If $f : X \to Y$ is a regular map between irreducible varieties $X$ and $Y$ then $f(X)$ is a finite union of locally closed subsets of $Y$.*

Therefore, $G \cdot x = \bigcup_{i=1}^n U_i \cap C_i$ where $U_i$ are open in $X$ and $C_i$ are closed in $X$. Let $U := \bigcap_{i=1}^n U_i$. If $y \in \overline{G \cdot x} \cap U$ then $y \in \bigcup_{i=1}^n \overline{U_i \cap C_i}$, and so $y \in \overline{C_i} = C_i$ for some $i$. So $y \in U_i \cap C_i \subset G \cdot x$. Thus, $\overline{G \cdot x} \cap U \subset G \cdot x$. Therefore, $G(\overline{G \cdot x} \cap U) = G \cdot x$, but this is an open subset of $\overline{G \cdot x}$. So $G \cdot x = \overline{G \cdot x} \cap V$ for some $V \subset X$ open. Thus, $G \cdot x$ is locally closed.

To prove the second part, apply the following theorem [Sha94, Section 6.3 Theorem 7].

*If $f : X \to Y$ is a regular map where $X$ and $Y$ are irreducible varieties and $f(X) = Y$ then there is a nonempty open $U \subset Y$ such that for every $y \in U$,* $\dim f^{-1}(y) = \dim(X) - \dim(Y)$.

If we apply this to the action map $G \to G \cdot x$, it says that there is an open set on which the dimension of the fibres equals $\dim(G) - \dim(G \cdot x)$. But in fact all the fibres are isomorphic to $\mathrm{stab}_G(x)$ (this is one proof that $\mathrm{stab}_G(x)$ is an algebraic set) and so we get the desired equality. □

### 14.1. **Back to our situation.** $X = \mathrm{Rep}(Q, \alpha)$ and $G = GL(\alpha)$. We prove the following theorem.

**Theorem 14.2.** *If $X \in \mathrm{Rep}(Q, \alpha)$ then*

$$\dim(\mathrm{Rep}(Q, \alpha)) - \dim(G \cdot X) = \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}Q}(X) - q(\alpha).$$

To explain the statement: on the left, dim means the algebro-geometric dimension. On the right $\dim_{\mathbb{C}}$ means the dimension as a vector space (although this equals the dimension of a vector space regarded as a variety). The number $q(\alpha)$ is the Tits form of the quiver evaluated at $\alpha$.

*Proof.* We apply the previous theorem to get

$$\dim(G \cdot X) = \dim(\mathrm{Rep}(Q, \alpha)) - \dim(\mathrm{stab}_G(X))$$

where $G = GL(\alpha)$. The group $\mathrm{stab}_G(X)$ consists of those endomorphisms of $X$ which have nonzero determinant. This is an open subset of the vector space of all endomorphisms of $X$, and therefore its dimension equals

$\dim \mathrm{End}_{\mathbb{C}Q}(X)$. Also, we have seen that $\dim(\mathrm{Rep}(Q,\alpha)) = \sum_{a:i\to j} \alpha_i\alpha_j$, and clearly $\dim(GL(\alpha)) = \sum_i \alpha_i^2$. The definition of $q$ now yields the desired result. $\square$

Recall that $q : \mathbb{Q}^n \to \mathbb{Q}$ is *positive definite* if $q(\alpha) \geq 0$ for all $\alpha \in \mathbb{Q}^n$, and $q(\alpha) = 0$ if and only if $\alpha = 0$.

**Lemma 14.3.** *If $Q$ is a quiver without cycles and $\mathbb{C}Q$ is of finite type, then $q$ is a positive definite form.*

*Proof.* If $\mathbb{C}Q$ has finite type, then for each $\alpha \in \mathbb{Z}^n_{\geq 0}$, there are finitely many isomorphism classes of modules of dimension vector $\alpha$, by the Krull-Schmidt Theorem. Therefore, there are only finitely many orbits of $GL(\alpha)$ on $\mathrm{Rep}(Q,\alpha)$. We always have $\mathrm{Rep}(Q,\alpha) \neq \varnothing$, since you can just take a representation with all maps being 0. So we have

$$\mathrm{Rep}(Q,\alpha) = \bigsqcup G \cdot X,$$

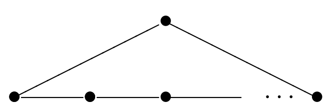
the disjoint union of finitely many orbits. Thus, there must be one orbit $G \cdot X$ with $\dim(G \cdot X) = \dim(\mathrm{Rep}(Q,\alpha))$. Theorem 14.2 now yields $q(\alpha) = \dim(\mathrm{End}_{\mathbb{C}Q}(X)) \geq 1$. Now if $\lambda \in \mathbb{Q}^n$ then there exists $m \in \mathbb{Z}$ with $m|\lambda| \in \mathbb{Z}^n_{\geq 0}$, where $|\lambda|$ denotes the vector with $|\lambda|_i = |\lambda_i|$ for each $i$. So $q(m|\lambda|) \geq 1$. But

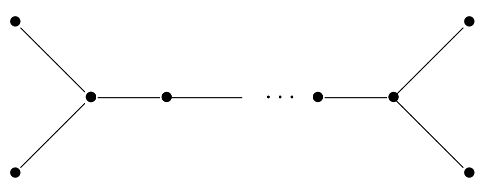
$$q(m|\lambda|) = m^2 \sum \lambda_i^2 - \sum_{a:i\to j} |\lambda_i||\lambda_j| \leq m^2 q(\lambda),$$

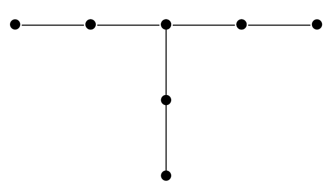
so $q(\lambda) \geq 1/m^2 > 0$. $\square$

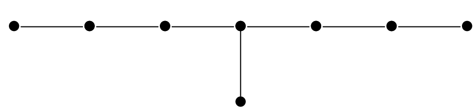
Therefore, all we need to do is classify all those graphs for which the quadratic form is positive definite. This is what we will do in the rest of this lecture. The argument is pure graph theory and requires no algebra or geometry.

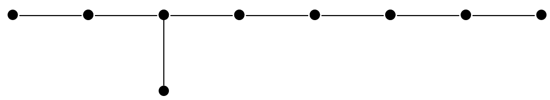
**Lemma 14.4.** *If $\Gamma$ is a connected graph, maybe with loops, then either $\Gamma$ is ADE, or $\Gamma$ contains one of the following graphs as a subgraph.*

$\widetilde{A_n}, n \geq 0$ :


$\widetilde{D_m}, m \geq 4$ :


$\widetilde{E_6}$ :


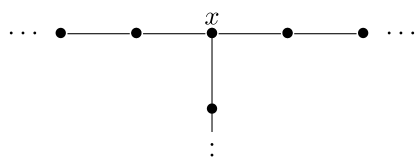
$\widetilde{E_7}$ :


$\widetilde{E_8}$ : 

*Here, if $X \in \{A, D, E\}$ then $\widetilde{X_k}$ has $k+1$ vertices. The graph $\widetilde{A_0}$ is by definition one vertex with a loop. The graph $\widetilde{A_1}$ consists of two vertices joined by a pair of edges.*

*The graphs in the list are called* Euclidean *or* extended Dynkin *graphs.*

*Proof.* If $\Gamma$ contains loops or multiple edges then $\Gamma$ contains $\widetilde{A_0}$ or $\widetilde{A_1}$. If $\Gamma$ has a cycle then $\Gamma$ contains some $\widetilde{A_n}$. If not, then $\Gamma$ is a tree. Take a longest path in $\Gamma$. If there are no branches off this path, then $\Gamma$ is $A_n$. If there are $\geq 2$ branches then $\Gamma$ contains $\widetilde{D_n}$. Otherwise, there is exactly one branch, and $\Gamma$ either contains $\widetilde{D_n}$, or consists of three paths joined at a vertex $x$. So $\Gamma$ looks like




Now either $\Gamma$ is $D_4$ or $D_5$ or contains $\widetilde{E_6}$, or else one branch from $x$ has length 1 and the others have length at least 2. Then either $\Gamma$ is $E_6$ or else $\Gamma$ contains $\widetilde{E_7}$, or else the branches at $x$ have length 1, 2 and some number $\geq 3$, and so on. $\square$

Notice that the theorem does not claim there is anything magical about the $ADE$ graphs. We could have taken some other set of graphs and extensions of them and proved exactly the same theorem.

**Definition 14.5.** *If $\Gamma$ is a graph with no loops, we define $n_{ij}$ to be the number of edges between $i$ and $j$, where $i, j \in \{1, 2, \ldots, n\}$, and where $\{1, 2, \ldots, n\}$ is the vertex set of $\Gamma$. We define the* quadratic form *$q_\Gamma : \mathbb{Q}^n \to \mathbb{Q}$ of $\Gamma$ by*

$$q_\Gamma(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum_{i<j} n_{ij}\alpha_i\alpha_j.$$

*If $\Gamma$ is the underlying graph of a quiver $Q$ then $q_\Gamma$ is the Tits form of $Q$.*

*For $\alpha, \beta \in \mathbb{Q}^n$ we define*

$$(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta).$$

*This may also be written as*

$$(\alpha, \beta) = 2\sum_{i=1}^{n} \alpha_i\beta_i - \sum_{i<j} n_{ij}(\alpha_i\beta_j + \beta_i\alpha_j).$$

*The form $(-,-)$ is called the* bilinear form *of $\Gamma$.*

**Theorem 14.6.** *If $\Gamma$ is a connected graph and there is a vector $\beta \neq 0$ with $\beta_i \geq 0$ for all $i$ and $(\beta, -) = 0$, then $\beta_i > 0$ for all $i$ and $q_\Gamma$ is a positive semidefinite form. Furthermore, $q_\Gamma(\alpha) = 0$ if and only if $\alpha \in \mathbb{Q}\beta$.*

*Proof.* Taken from [CB, Section 4].

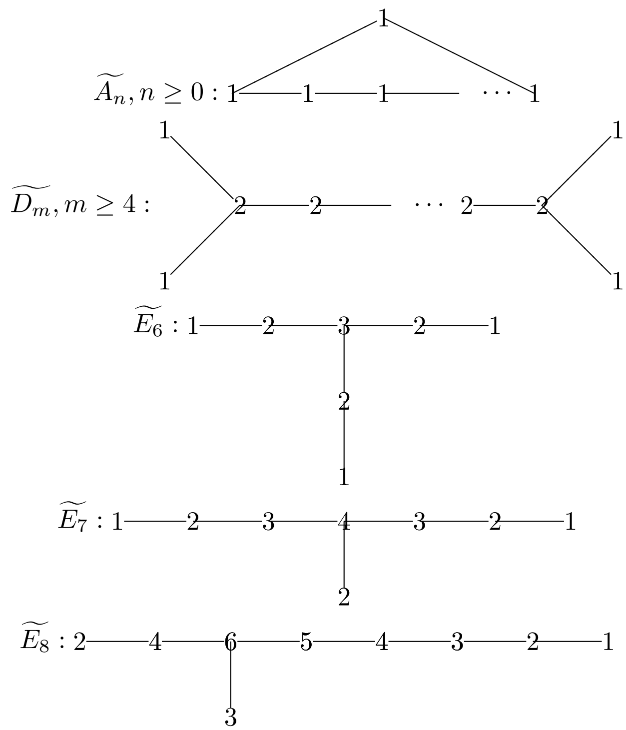
If $\beta_i = 0$ then if $\alpha_i$ is defined to be the standard basis vector with $(\alpha_i)_j = \delta_{ij}$, we have $(\beta, \alpha_i) = -\sum n_{ij}\beta_j = 0$ and so $\beta_j = 0$ for all $j$ such that there is an edge between $i$ and $j$. Since $\Gamma$ is connected, we get $\beta = 0$, a contradiction. So $\beta_i > 0$ for all $i$.

Now, for all $\alpha \in \mathbb{Q}^n$, we have

$$q_\Gamma(\alpha) = \sum_{i<j} n_{ij} \frac{\beta_i \beta_j}{2} \left( \frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} \right)^2$$

(the calculation of this is given in [CB]). In particular, $q_\Gamma$ is positive semidefinite. If $q_\Gamma(\alpha) = 0$ then $\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} = 0$ for all $i$ and $j$, so if we let $\zeta$ be the common value of $\alpha_i/\beta_i$, then $\alpha = \zeta\beta$. Therefore, $\alpha \in \mathbb{Q}\beta$. $\square$

For each Euclidean graph, we exhibit a $\beta$ with $\beta_i > 0$ for all $i$ and $(\beta, -) = 0$. It can be checked that the given $\beta$ has these properties via an explicit calculation (it is enough to show $(\beta, \alpha_i) = 0$ for each standard basis vector $\alpha_i$).




Therefore, $q_\Gamma$ is positive semidefinite for each Euclidean graph $\Gamma$.

Now if $\Gamma$ is ADE then $\Gamma \subsetneq \Gamma'$ with $\Gamma'$ Euclidean and $\Gamma'$ having one more vertex than $\Gamma$. The value of $q_\Gamma$ at $\alpha \in \mathbb{Q}^n$ is the value of $q_{\Gamma'}$ at the vector $\alpha$ extended by 0 at the extra vertex of $\Gamma'$. So $q_\Gamma$ is positive semidefinite and $q_\Gamma(\alpha) = 0$ implies $\alpha \in \mathbb{Q}\beta$ which implies $\alpha = 0$ since $\alpha_e = 0$ where $e$ is the extra vertex of $\Gamma'$. So $q_\Gamma$ is positive definite.

Conversely, suppose $q_\Gamma$ is positive definite. Suppose $\Gamma$ is not ADE. Then $\Gamma \supsetneq \Gamma'$ with $\Gamma'$ Euclidean, by Lemma 14.4. If $\Gamma'$ contains all the vertices of $\Gamma$, then $q_\Gamma(\beta) < q_{\Gamma'}(\beta) = 0$, a contradiction to $q_\Gamma$ positive definite. Therefore, there is a vertex $k$ of $\Gamma$ not in $\Gamma'$. Take $\alpha := 2\beta + \alpha_k$ where $\alpha_k$ is the standard basis vector, and $\beta$ is regarded as a vector in $\mathbb{Q}^n$ (where $n$ is the number of vertices of $\Gamma$) via extension by zero.

Then an explicit check using the definition of $q_\Gamma$ shows that $q_\Gamma(\alpha) < 4q_{\Gamma'}(\beta) = 0$, which again contradicts that $q_\Gamma$ is positive definite. Thus $\Gamma$ is ADE.

**Theorem 14.7.** *If $\Gamma$ is a connected graph with no loops then $q_\Gamma$ is positive definite if and only if $\Gamma$ is ADE.*

**Corollary 14.8** (Half of Gabriel's Theorem)**.** *If $Q$ is a quiver and $\mathbb{C}Q$ is finite-dimensional and of finite type then the underlying graph of $Q$ is ADE.*

## 15. Lecture 15

In the last lecture, we showed that if $Q$ is a quiver without oriented cycles, then if $\mathbb{C}Q$ has finite type, then $q$ is positive definite. We then showed that $q$ is positive definite if and only if the underlying (connected) graph is ADE. This proves half of Gabriel's Theorem.

Now we want to show that every quiver whose underlying graph is ADE has finite type. Again, the proof is copied from [CB].

**Lemma 15.1.** *Suppose $X \in \mathrm{Rep}(Q, \alpha)$. Then*

$$\mathrm{dimEnd}_{\mathbb{C}Q}(X) - q(\alpha) = \mathrm{dimExt}^1_{\mathbb{C}Q}(X, X).$$

*Proof.* Recall the *standard resolution* of $X$.

$$0 \to \bigoplus_{a \in Q_1} Ae_{h(a)} \otimes_{\mathbb{C}} e_{t(a)}X \to \bigoplus_{i \in Q_0} Ae_i \otimes_{\mathbb{C}} e_iX \to X \to 0$$

This sequence is exact. Apply $\mathrm{Hom}(-, X)$ and use the long exact Ext–sequence. This gives a sequence

$$0 \to \mathrm{Hom}(X, X) \to \mathrm{Hom}(\bigoplus_{i \in Q_0} Ae_i^{\dim(e_iX)}, X) \to \mathrm{Hom}(\bigoplus_{a \in Q_1} Ae_{h(a)}^{\dim(e_{t(a)}X)}, X) \to \mathrm{Ext}^1(X, X) \to 0$$

The alternating sum of the dimensions of the terms is zero, because the sequence is exact. This yields

$$\mathrm{dimEnd}(X) - \sum_{i \in Q_0} \dim(e_iX)\mathrm{dimHom}(Ae_i, X) + \sum_{a \in Q_1} \dim(e_{t(a)}X)\mathrm{dimHom}(Ae_{h(a)}, X) - \mathrm{dimExt}^1(X, X) = 0.$$

Now using the isomorphism of vector spaces $\mathrm{Hom}(Ae, X) \cong eX$ gives the desired result, because if $X$ is regarded as a representation of $Q$, then the dimension vector of $X$ is $\alpha = (\dim(e_iX))_{i \in Q_0}$. $\square$

We aim to show that if $X$ is indecomposable then $q(\underline{\dim}(X)) = 1$.

**Definition 15.2.** *If $A$ is an algebra and $M$ is an $A$–module then $M$ is called a* brick *if* $\mathrm{dimEnd}_A(M) = 1$.

Fitting's Lemma shows that every brick is indecomposable. Not every indecomposabale module is a brick, for example if $A = \mathbb{C}[x]/(x^2)$ then the $A$–module $A$ is not a brick.

The following key lemma is due to Ringel. The lemma and its proof are taken from [CB, Section 2, Lemma 2].

**Lemma 15.3.** *Let $A$ be a hereditary $\mathbb{C}$–algebra and $X$ an indecomposable $A$–module which is not a brick. Then $X$ has a submodule $U$ such that $U$ is a brick and* $\mathrm{Ext}^1(U,U) \neq 0$.

*Proof.* We show that there exists $U \subsetneq X$ such that $U$ is indecomposable and $\mathrm{Ext}^1(U,U) \neq 0$. If $U$ is not a brick, we can then repeat and use induction on $\dim(X)$ to get the result.

Since $X$ is not a brick, by Fitting's Lemma there exists some $\theta \in \mathrm{End}(X)$ such that $\theta$ is not an isomorphism. Indeed, given a $\theta \in \mathrm{End}(X)$ which is not a scalar multiple of the identity, pick an eigenvalue $\lambda$ of $\theta$. Then $\theta - \lambda 1$ is a nonzero nonisomorphism.

Now choose $\theta \in \mathrm{End}(X)$ such that $\mathrm{Im}(\theta)$ has the smallest possible dimension among $\theta$ with $\mathrm{Im}(\theta) \neq 0$. Then $\mathrm{Im}(\theta) \neq X$ because we have already shown that there is some endomorphism of $X$ which is not surjective. So $\theta$ must be nilpotent, by Fitting's Lemma.

Now $\theta^2$ is another endomorphism of $X$ and $\mathrm{Im}(\theta^2) \subset \mathrm{Im}(\theta)$. If this is an equality, then $\mathrm{Im}(\theta^n) = \mathrm{Im}(\theta)$ for all $n$, which is impossible since $\theta$ is nilpotent. Therefore, $\mathrm{Im}(\theta^2) \subsetneq \mathrm{Im}(\theta)$ and we get $\theta^2 = 0$ by choice of $\theta$.

Let $\{K_j\}$ be the indecomposable summands of $\ker(\theta)$. Then there is some $j$ such that the composition

$$\alpha : \mathrm{Im}(\theta) \hookrightarrow \ker(\theta) \twoheadrightarrow K_j$$

is nonzero. Define $U := K_j$. Then $U$ is indecomposable, so we need to check that $\mathrm{Ext}^1(U,U) \neq 0$. We first show that $\ker(\alpha) = 0$. Indeed, $\mathrm{Im}(\alpha)$ is also the image of the composition

$$X \xrightarrow{\theta} \mathrm{Im}(\theta) \xrightarrow{\alpha} K_j \longrightarrow X$$

and if $\ker(\alpha) \neq 0$, then this has dimension strictly less than $\dim(\mathrm{Im}(\theta))$, which contradicts the choice of $\theta$. Therefore, $\ker(\alpha) = 0$.

Now we have a short exact sequence

$$0 \to \mathrm{Im}(\theta) \to K_j \to \mathrm{cok}(\alpha) \to 0$$

Applying the long exact Ext–sequence and using $\mathrm{Ext}^2 = 0$, we get an exact sequence

$$\mathrm{Ext}^1(\mathrm{cok}(\alpha), K_j) \to \mathrm{Ext}^1(K_j, K_j) \to \mathrm{Ext}^1(\mathrm{Im}(\theta), K_j) \to 0$$

and so it suffices to show that $\mathrm{Ext}^1(\mathrm{Im}(\theta), K_j) \neq 0$. Suppose $\mathrm{Ext}^1(\mathrm{Im}(\theta), K_j) = 0$. Then the following short exact sequence splits (here we are using a property of $\mathrm{Ext}^1$, namely that $\mathrm{Ext}^1(X,Y) = 0$ if and only if every short exact sequence $0 \to Y \to Z \to X \to 0$ is split. This can be proved using the long exact sequence for Ext.)

$$0 \to K_j \to K_j \oplus X / \{(-\pi_j(x), x) : x \in \ker(\theta)\} \to \mathrm{Im}(\theta) \to 0.$$

In the above sequence, the first map is $k \mapsto (k, 0)$ and the second map is $(k, x) \mapsto \theta(x)$. The middle term is the quotient of $K_j \oplus X$ by the submodule $\{(-\pi_j(x), x) : x \in \ker(\theta)\}$, where $\pi_j : \ker(\theta) \to K_j$ denotes the projection. It is easy to check that the sequence is well-defined and exact by a direct calculation. The middle term is the pushout of the diagram

$$\begin{tikzcd} \ker(\theta) \arrow[r] \arrow[d, "\pi_j"'] & X \\ K_j & \end{tikzcd}$$

which does not come out of nowhere, but is a standard construction in homological algebra (see [Rot09]).

Anyway, assuming $\mathrm{Ext}^1(\mathrm{Im}(\theta), K_j) = 0$, the above sequence splits, and so there is a map $\lambda : K_j \oplus X/\{(-\pi_j(x), x) : x \in \ker(\theta)\} \to K_j$ such that $\lambda(k, 0) = k$ for $k \in K_j$. Denote by $\gamma : X \to K_j \oplus X/\{(-\pi_j(x), x) : x \in \ker(\theta)\}$ the map $\gamma : x \mapsto (0, x)$ and denote by $\iota$ the inclusion of $K_j$ in $X$. Then for $k \in K_j$, $\lambda\gamma\iota(k) = \lambda(0, k) = \lambda(\pi_j(k), 0) = \lambda(k, 0) = k$. Therefore, the sequence

$$0 \longrightarrow K_j \xrightarrow{\iota} X \longrightarrow \mathrm{cok}(\iota) \longrightarrow 0$$

splits. It follows that $X$ is decomposable, a contradiction. □

**Definition 15.4.** *If $\Gamma$ is an ADE or Euclidean graph with vertex set $\{1, 2, \ldots, n\}$, then $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a* root *if $q_\Gamma(\alpha) \leq 1$. If $Q$ is a quiver then a* root *of $Q$ is a root of the underlying graph of $Q$.*

The following theorem and its elegant proof come from [CB, Section 5].

**Theorem 15.5.** *If $Q$ is a quiver with underlying graph ADE, then there is an injection from the set of isomorphism classes of indecomposable $\mathbb{C}Q$–modules to the set of roots, given by $X \mapsto \underline{\dim}(X)$.*

*Proof.* Let $X$ be an indecomposable $\mathbb{C}Q$–module. If $X$ is not a brick, then by Lemma 15.3, there exists $U \subset X$ such that $U$ is a brick and $\mathrm{Ext}^1(U, U) \neq 0$. But then by Lemma 15.1, we have $q(\underline{\dim}(U)) = 1 - \dim\mathrm{Ext}^1(U, U) \leq 0$, which contradicts that $q$ is positive definite. So $X$ is a brick. Therefore, by Lemma 15.1 again, we have $q(\underline{\dim}(X)) = 1 - \dim\mathrm{Ext}^1(X, X) > 0$. So $\mathrm{Ext}^1(X, X) = 0$ and $q(\underline{\dim}(X)) = 1$, so $\underline{\dim}(X)$ is a root.

Now, if $X, Y$ are indecomposable and $\underline{\dim}(X) = \underline{\dim}(Y) =: \alpha$, consider the orbits $G \cdot X$ and $G \cdot Y$ in $\mathrm{Rep}(Q, \alpha)$. Recall that

$$\dim\mathrm{Rep}(Q, \alpha) - \dim(G \cdot X) = \dim\mathrm{End}_{\mathbb{C}Q}(X) - q(\alpha).$$

Since $q(\alpha) = 1$, we get that $\dim(G \cdot X) = \dim\mathrm{Rep}(Q, \alpha)$. So $\overline{G \cdot X} = \mathrm{Rep}(Q, \alpha)$ by our definition of dimension. But $G \cdot X$ is open in $\overline{G \cdot X}$, hence is open in $\mathrm{Rep}(Q, \alpha)$. Similarly, $G \cdot Y$ is open in $\mathrm{Rep}(Q, \alpha)$. But $\mathrm{Rep}(Q, \alpha)$ is just a copy of $\mathbb{C}^N$, so is irreducible. Therefore, $G \cdot X \cap G \cdot Y \neq \varnothing$. So $G \cdot X = G \cdot Y$ and $X$ and $Y$ are in the same orbit, so are isomorphic. This proves that $X \mapsto \underline{\dim}(X)$ is an injection. □

## 16. Lecture 16

To finish the proof of Gabriel's Theorem, we just need to show that the set of roots of an ADE graph is finite. Note that in the ADE case, $\alpha$ is a root if and only if $q(\alpha) = 1$. Recall also that if $\Gamma$ is a Euclidean graph then $q_\Gamma$ is positive semidefinite and there is a $\beta \in \mathbb{Z}^n_{\geq 0}$ ($n$ = number of vertices of $\Gamma$) such that $q_\Gamma(\alpha) = 0$ if and only if $\alpha \in \mathbb{Q}\beta$. We use the following lemma.

**Lemma 16.1.** *If $\alpha$ is a root of a Euclidean graph $\Gamma$ then either $\alpha_i \geq 0$ for all $i$ or $\alpha_i \leq 0$ for all $i$.*

*Proof.* Suppose $\Gamma$ is Euclidean and $\alpha$ is a root. Write $q = q_\Gamma$. Write $\alpha = \alpha_+ + \alpha_-$ with $(\alpha_+)_i \geq 0$ for all $i$, $(\alpha_-)_i \leq 0$ and $(\alpha_+)_i(\alpha_-)_i = 0$. Then since $\alpha$ is a root, $q(\alpha) = q(\alpha_+ + \alpha_-) \leq 1$. So we have

$$q(\alpha_+) + q(\alpha_-) + (\alpha_+, \alpha_-) \leq 1.$$

Now, by definition of $(-,-)$, $(\alpha_+, \alpha_-) \geq 0$ and so

$$q(\alpha_+) + q(\alpha_-) \leq 1.$$

It follows that either $q(\alpha_+) = 0$ or $q(\alpha_-) = 0$. Say $q(\alpha_+) = 0$. Then either $\alpha_+ = 0$ or $\alpha_+$ is a multiple of $\beta$ and so $(\alpha_+)_i \neq 0$ for all $i$, whence $\alpha_- = 0$. So either $\alpha = \alpha_+$ or $\alpha = \alpha_-$ as required. Similarly, the same conclusion is reached if $q(\alpha_-) = 0$. $\square$

**Lemma 16.2.** *If $\Gamma$ is an ADE graph then the set of roots of $\Gamma$ is finite.*

*Proof.* Suppose $\Gamma$ has $n$ vertices. Let $\Gamma \subsetneq \Gamma'$ with $\Gamma'$ Euclidean and $\Gamma'$ having one extra vertex $e \notin \Gamma$. Let $\alpha$ be a root of $\Gamma$. Then regard $\alpha$ as a vector in $\mathbb{Z}^{n+1}_{\geq 0}$ by putting $\alpha_e = 0$. We then have $q_{\Gamma'}(\alpha \pm \beta) = 1$. So $\alpha + \beta$ and $\alpha - \beta$ are roots of $\Gamma'$. But $(\alpha + \beta)_e > 0$ and so by the previous lemma, $\alpha_i \geq -\beta_i$ for all $i$. Also, $(\alpha - \beta)_e < 0$ and so $\alpha_i \leq \beta_i$ for all $i$. Therefore, $-\beta_i \leq \alpha_i \leq \beta_i$ for all $i$ and it follows that the number of possible $\alpha$ is finite. $\square$

**Corollary 16.3.** *If $Q$ is a quiver without cycles whose underlying graph is ADE then $\mathbb{C}Q$ is of finite type.*

This completes the proof of Gabriel's Theorem.

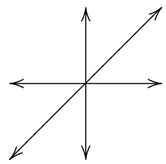
### 16.1. Roots.

Let $Q$ be an ADE quiver. Then $R = \{\alpha : q(\alpha) = 1\}$ is a *root system* in $\mathbb{Q}^n$. That is, it satisfies the following axioms.

1. $R$ spans $\mathbb{Q}^n$ and is a finite set.
2. $\alpha \in R \implies -\alpha \in R$ but $k\alpha \notin R$ if $k \neq \pm 1$.
3. $\alpha, \beta \in R \implies \alpha - 2\frac{(\alpha,\beta)}{(\beta,\beta)}\beta \in R$.
4. $2\frac{(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

Let $Q = \bullet \longrightarrow \bullet$ . Then the roots are $\pm(0,1)$, $\pm(1,0)$, $\pm(1,1)$.

If we graph the elements of $R$, they look like this.



If you have taken a Lie algebras course, this might not look like one of the familiar root systems. However, this is only because the inner product we are using on $\mathbb{Q}^2$ is not the usual one. After an appropriate change of coordinates, this is in fact the root system of type $A_2$.

Further analysis (see [CB]) shows that for an ADE quiver, there is a bijection between isomorphism classes of indecomposable modules and positive roots, given by $X \mapsto \underline{\dim}(X)$. In [CB], the same thing is proved for Euclidean graphs as well. For example, if $Q = \bullet \rightrightarrows \bullet$ , then the quadratic form is $q(\alpha_0, \alpha_1) = (\alpha_0 - \alpha_1)^2$ and the dimension vectors of indecomposable representations are $(a+1, a+1)$, $(a+1, a)$ and $(a, a+1)$, $a \geq 0$.

The ultimate version of the theorem is *Kac' Theorem*, which we now describe without proof.

16.2. **Kac' Theorem.** Let $\Gamma$ be a connected graph without vertex loops. Let the vertices of $\Gamma$ be $\{1, 2, \ldots, n\}$. For $i < j$, let $n_{ij}$ be the number of edges between $i$ and $j$. Define the form $q_\Gamma$ as before, and the associated bilinear form by $(\alpha_i, \alpha_i) = 2$, $(\alpha_i, \alpha_j) = -n_{ij}$ where $i \neq j$. Here, $\alpha_i$ is the standard basis vector with $j^{th}$ entry $(\alpha_i)_j = \delta_{ij}$.

For $1 \leq i \leq n$, define $r_i : \mathbb{Q}^n \to \mathbb{Q}^n$ by

$$r_i(\lambda) = \lambda - 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i$$

then $r_i(\mathbb{Z}^n) \subset \mathbb{Z}^n$, and $r_i$ is a reflection, that is, $r_i(\alpha_i) = -\alpha_i$ and $r_i(\lambda) = \lambda$ if $(\lambda, \alpha_i) = 0$, and $r_i^2 = 1$.

**Definition 16.4.** *The* Weyl group *of* $\Gamma$ *is the subgroup of* $GL(\mathbb{Q}^n)$ *generated by* $r_1, r_2, \ldots, r_n$.

Write $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and define

$$M := \{\alpha \in \mathbb{Z}^n_{\geq 0} \setminus \{0\} : (\alpha, \alpha_i) \leq 0 \text{ for all } i\}$$

**Definition 16.5.** *The* root system *of* $\Gamma$ *is the set*

$$\Delta(\Gamma) = W(\Pi) \cup W(M) \cup W(-M).$$

Here is one version of Kac' Theorem (not the strongest possible statement).

**Theorem 16.6** (Kac). *Let $Q$ be a connected quiver over $\mathbb{C}$ with no oriented cycles. Let $\alpha \in \mathbb{Z}^n$.*

(1) *There exists an indecomposable representation of dimension $\alpha$ if and only if $\alpha \in \Delta(\Gamma) \cap \mathbb{Z}^n_{\geq 0}$.*
(2) *There exists a* unique *indecomposable representation of dimension $\alpha$ if and only if $\alpha \in W(\Pi)$.*

In the case of an ADE quiver, $M = \varnothing$ and $\Delta(\Gamma) = W(\Pi)$. There is an indecomposable representation of dimension $\alpha$ if and only if $q(\alpha) = 1$. We have seen that the set of roots is finite, and it is clearly invariant under $W$. Kac' Theorem asserts that in fact every $\alpha$ with $q(\alpha) = 1$ is in the orbit of $\Pi$ under $W$. In the Euclidean case, $M = \mathbb{Z}_{\geq 0}\beta \setminus \{0\}$ and $q(\alpha) = 0$ if and only if $\alpha \in W(M) \cup W(-M) \cup \{0\}$.

The proof of the theorem is quite difficult and involves a reduction to characteristic $p$ and counting over finite fields. The proof can be found in [Kac83b].

We now explain the meaning of the term "root system" and the connection to Lie algebras. We don't give the definition of Lie algebras since everybody in the class knew it already. Given a graph $\Gamma$ as above, we define a $\mathbb{C}$–Lie algebra $\widetilde{\mathfrak{g}}(\Gamma)$ via generators and relations. The generators of $\widetilde{\mathfrak{g}}(\Gamma)$ consist of a vector space $\mathfrak{h}$ with basis $\alpha_1, \ldots, \alpha_n$, together with symbols $e_i, f_i$, $1 \leq i \leq n$. The form $(-,-)$ is extended to $\mathfrak{h}$ linearly. The relations are as follows.

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij}\alpha_j \\
&[h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h} \\
&[h, e_i] = (h, \alpha_i)e_i \quad \text{for } h \in \mathfrak{h} \\
&[h, f_i] = -(h, \alpha_i)f_i \quad \text{for } h \in \mathfrak{h}
\end{aligned}
$$

The relations say that $e_i$ and $f_i$ are eigenvectors for the action of $\mathfrak{h}$ with eigenvalue $(\alpha_i, -)$, $-(\alpha_i, -)$ respectively. Although it needs to be proved carefully, it is intuitively quite clear that we should have the following *root space decomposition*

$$
\widetilde{\mathfrak{g}}(\Gamma) = \left( \bigoplus_{\alpha \in \sum \mathbb{Z}_{\geq 0}\alpha_i \setminus \{0\}} \widetilde{\mathfrak{g}}(\Gamma)_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \sum \mathbb{Z}_{\geq 0}\alpha_i \setminus \{0\}} \widetilde{\mathfrak{g}}(\Gamma)_{\alpha} \right)
$$

where

$$
\widetilde{\mathfrak{g}}(\Gamma)_\alpha := \{x \in \widetilde{\mathfrak{g}}(\Gamma)_\alpha : [h, x] = (\alpha, h)x \text{ for all } h \in \mathfrak{h}\}.
$$

Fact: there is a unique ideal $\mathfrak{a}$ which is the largest ideal with the property that $\mathfrak{a} \cap \mathfrak{h} = 0$.

**Definition 16.7.** *The* Kac-Moody algebra *of* $\Gamma$ *is*

$$
\mathfrak{g}(\Gamma) := \widetilde{\mathfrak{g}}(\Gamma)/\mathfrak{a}.
$$

Fact: there is still a root space decomposition $\mathfrak{g}(\Gamma) = \bigoplus \mathfrak{g}_\alpha$. We say $\alpha$ is a *root* if $\mathfrak{g}(\Gamma)_\alpha \neq 0$.

**Theorem 16.8.** *The roots of* $\mathfrak{g}(\Gamma)$ *are precisely the set* $\Delta(\Gamma)$ *defined above.*

For a proof of the above facts and the theorem, see [Kac83a, Sections 5.1, 5.2, Theorem 5.4].

The theorem indicates a deep connection between the representation theory of quivers and the representation theory of infinite-dimensional Lie algebras.

## 17. Lecture 17

17.1. **Auslander-Reiten Theory.** We are now going to begin a new topic: Auslander-Reiten theory. This is something you often see in talks about finite-dimensional algebras, and is much studied at the moment. Each finite-dimensional algebra has an *Auslander-Reiten quiver*, which is a (possibly infinite) directed graph which can be viewed as an approximation to the module category. Recall that our goal is to understand the module category of an algebra. To do this we have to write down all the modules and the homomorphisms between them and say how the homomorphisms compose. The Krull-Schmidt Theorem reduces this problem to only looking at the indecomposable modules, since maps between direct sums of modules can be viewed as matrices. However, all this data is still hard to calculate. The Auslander-Reiten quiver can be viewed as a kind of first approximation to this. Another way of looking at it is as follows: for any algebra $A$, there are certain exact sequences over $A$ which are non-split but very special. These sequences were discovered by Auslander and Reiten and are called *Auslander-Reiten sequences*. It turns out that they are of great theoretical importance. The Auslander-Reiten quiver is simply a way of writing down all the modules, but not the maps, in the Auslander-Reiten sequences over $A$.

Given an algebra $A$, the Auslander-Reiten quiver $Q$ is defined as follows.

(1) The vertices of $Q$ are the isomorphism classes of indecomposable $A$–modules.
(2) Given two isomorphism classes $X$ and $Y$, the number of arrows from $X$ to $Y$ is the dimension of the vector space

$$\mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$$

where

$$\mathrm{rad}(X,Y) := \{f : X \to Y : f \text{ is not an isomorphism}\}$$

and $\mathrm{rad}^2(X,Y)$ is the span of all the maps $gh : X \to Y$ where $h : X \to Z$, $g : Z \to Y$ for some $Z$, and $h \in \mathrm{rad}(X,Z)$, $g \in \mathrm{rad}(Z,Y)$ (note: we haven't defined $\mathrm{rad}(X,Z)$ yet for arbitrary $Z$).

It should not be obvious at this point that $\mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ is a vector space; we will prove this today.

(3) The Auslander-Reiten quiver has vertices, arrows and a third extra structure called the *Auslander-Reiten translate* $\tau$. This is very important, but we will not define it until later.

Here are some reasons why the Auslander-Reiten (or AR) quiver is important:

(1) It is often computable.
(2) It is useful in applications. For example, it was originally invented to prove the *First Brauer-Thrall conjecture*. This conjecture states that if $A$ is a finite-dimensional algebra and there is a number $N$ such that $\dim(X) < N$ for every indecomposable $A$–module $X$, then $A$ is of finite type. Unfortunately, we haven't seen enough examples to have a good idea of why such a statement might be true.

Most of the time, we will follow either [ASS06] or [ARS97] as indicated in the notes.

17.2. **The radical of a category.** To start with, we are going to generalise the notion of the Jacobson radical of a ring to categories. Recall that if $R$ is a ring, one of the definitions of the Jacobson radical of $R$ is

$$J(R) = \{x \in R : 1 - ax \text{ is a unit for all } a \in R\}.$$

**Definition 17.1.** *Let $A$ be a finite-dimensional algebra and let $X, Y$ be $A$–modules. We define*

$$\operatorname{rad}(X,Y) = \{f : X \to Y | \forall g : Y \to X, 1_X - gf \textit{ is an isomorphism}\}.$$

**Proposition 17.2.** *If $X$ and $Y$ are indecomposable then $f \in \operatorname{rad}(X,Y)$ if and only if $f$ is not an isomorphism.*

*Proof.* If $f$ is an isomorphism then $1_X - f^{-1}f = 0$ so $f \notin \operatorname{rad}(X,Y)$.

Conversely, if $f$ is not an isomorphism then let $g : Y \to X$. We show first that $gf$ is not an isomorphism. Suppose $gf$ is an isomorphism. Let $e = f(gf)^{-1}g \in \operatorname{End}(Y)$. Then $e^2 = e$ so $e$ is not nilpotent. By Fitting's Lemma, $e$ is an isomorphism. But $e^2 = e$ so $e = 1_Y$. Therefore, $(gf)^{-1}g$ is an inverse to $f$, which contradicts that $f$ is not an isomorphism. Thus, $gf$ is not an isomorphism.

By Fitting's Lemma, if $gf \in \operatorname{End}(X)$ is not an isomorphism, then it must be nilpotent. So $1_X - gf$ is a unit, which says that $f \in \operatorname{rad}(X,Y)$. □

**Lemma 17.3.** *If $f \in \operatorname{rad}(X,Y)$ and $g : Y \to Z$ and $h : W \to X$ then $gf \in \operatorname{rad}(X,Z)$ and $fh \in \operatorname{rad}(W,Y)$.*

*Proof.* For $gf$, it's obvious because if $v : Z \to X$ then $1_X - vgf$ is invertible by the assumption that $f \in \operatorname{rad}(X,Y)$.

For $fh : W \to Y$, let $u : Y \to W$. We want to show that $1_W - ufh$ is invertible. Since $f \in \operatorname{rad}(X,Y)$, $1_X - huf$ is invertible. Now you can check explicitly that $1_W + uf(1_X - huf)^{-1}h$ is a two-sided inverse to $1_W - ufh$, as required. □

**Proposition 17.4.** *For any modules $X, Y$, $\operatorname{rad}(X,Y)$ is a vector space.*

*Proof.* If $f \in \operatorname{rad}(X,Y)$ and $\lambda \in k$ ($k$ is the base field) then $\lambda f \in \operatorname{rad}(X,Y)$; this is obvious.

Now let $f_1, f_2 \in \operatorname{rad}(X,Y)$ and let $g : Y \to X$. We need to show that $1_X - g(f_1 + f_2)$ has an inverse. First, $1_X - gf_1$ has an inverse, say $t$. So $t(1_X - g(f_1 + f_2)) = t(1_X - gf_1) - tgf_2 = 1_X - tgf_2$. Now, by the previous lemma, $1_X - tgf_2$ has an inverse, say $t'$. So $t't(1_X - g(f_1 + f_2)) = 1_X$.

Next, $1_x - gf_2$ has an inverse $u'$ so $(1_X - g(f_1 + f_2))u' = 1_X - gf_1u'$. Now, $f_1u' \in \operatorname{rad}(X,Y)$ by the previous lemma, and so $1_X - gf_1u'$ has an inverse $u$. Therefore, $(1_X - g(f_1 + f_2))u'u = 1_X$. From this, together with $t't(1_X - g(f_1 + f_2)) = 1_X$, we conclude that $u'u = t't = (1_X - g(f_1 + f_2))^{-1}$ □

We have proved that $\operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$ is a well-defined vector space. Now we want to interpret its meaning in terms of irreducible maps.

17.3. **Irreducible maps.** If we want to make a picture of the module category, it is a good idea to try to get rid of the morphisms which are compositions of other morphisms, and just keep the ones which can't be decomposed any further. The motivation for the definition of an irreducible map is to look for a suitable definition of a map which can't be broken down as the composition of other maps. A first guess is:

**Wrong Definition 17.5.** *$f : X \to Y$ is irreducible if $f$ is not an isomorphism and if $f = f_1 f_2$ then either $f_1$ or $f_2$ is an isomorphism.*

Unfortunately, the set of such maps is empty. This is because every $f : X \to Y$ can be factorised as follows:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & X \oplus Z \arrow[ur, "{(f,0)}"'] & \end{tikzcd}$$

Here, $Z$ is arbitrary and $X \to X \oplus Z$ is the inclusion of $X$ as a direct summand, and the map $(f, 0)$ is defined by $(f, 0)(x, z) = f(x) + 0(z) = f(x)$. (The reason for the notation $(f, 0)$ is that it is customary to think of the elements of $X \oplus Z$ as column vectors, and $(f, 0)$ as a $1 \times 2$ matrix.)

Also, every $f : X \to Y$ can be factorised as follows,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & Y \oplus Z \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

where the map $X \to Y \oplus Z$ is $x \mapsto (f(x), 0)$, and $\pi_Y$ is the projection. Again, $Z$ can be arbitrary.

Thus, Wrong Definition 17.5 has to be replaced by something which rules out the two factorisations above.

**Definition 17.6.** *We say:*

$a : X \to Y$ *is a* split monomorphism *if there exists* $b : Y \to X$ *with* $ba = 1_X$.

$a : X \to Y$ *is a* split epimorphism *if there exists* $b : Y \to X$ *with* $ab = 1_Y$.

By the familiar characterisations of split exact sequences, a split monomorphism is just the inclusion of a direct summand, possibly pre- and post-composed with isomorphisms, and a split epimorphism is just the projection onto a direct summand, possibly pre- and post-composed with isomorphisms.

**Definition 17.7.** $f : X \to Y$ *is* irreducible *if $f$ is not split mono or split epi and if $f = hg$ then either $g$ is split mono or $h$ is split epi.*

**Examples 17.8.** A non-example:

- If $S$ is simple then by Schur's Lemma there are no irreducible maps $S \to S$, because an irreducible map can neither be 0 nor an isomorphism.

- If $P$ is an indecomposable projective and $\mathrm{rad}(P) \neq 0$ then the inclusion map $i : \mathrm{rad}(P) \to P$ is irreducible. Prove this as an exercise (I will do it next time).

## 18. Lecture 18

We left off with the statement that if $P$ is an indecomposable projective with $\mathrm{rad}(P) \neq 0$ then $i : \mathrm{rad}(P) \to P$ is an irreducible map. Recall that $\mathrm{rad}(P)$ is the unique maximal submodule of $P$. To show that $i$ is irreducible, first we note that $i$ is not split epi because $i$ is not an epimorphism, and $i$ is not split mono because $P$ is an indecomposable module. Now suppose there is a factorisation $i = hg$.

$$\begin{tikzcd} \mathrm{rad}(P) \arrow[rr, "i"] \arrow[dr, "g"'] & & P \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

Consider $h(Z)$. If $h(Z) = P$ then $0 \to \ker(h) \to Z \to P \to 0$ is a short exact sequence and it splits because $P$ is projective. So $h$ is split epi. If not, then $h(Z)$ is a proper submodule of $P$ and so $h(Z) \subset \mathrm{rad}(P)$. So for $x \in \mathrm{rad}(P)$, $hg(x) = ix = x$ and therefore $g$ is split mono.

Another fact is that irreducible maps are either injective or surjective. If $f : X \to Y$ is irreducible then $f$ may be factorised as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\overline{f}"'] & & Y \\ & \mathrm{im}(f) \arrow[ur, "i"'] & \end{tikzcd}$$

where $i$ is the inclusion. Then either $i$ is surjective in which case $f$ is surjective, or else $\overline{f}$ is injective, in which case $f$ is injective.

The examples suggest that irreducible maps are quite scarce (as you will see if you try to think up further examples of them). Now we relate them to $\mathrm{rad}/\mathrm{rad}^2$.

**Theorem 18.1.** *Let $X$ and $Y$ be indecomposable. Then $f : X \to Y$ is irreducible if and only if $f \in \mathrm{rad}(X,Y) \setminus \mathrm{rad}^2(X,Y)$ (where $\setminus$ denotes the set difference).*

*Proof.* Suppose $f : X \to Y$ is irreducible. Then $f$ is not an isomorphism and so $f \in \mathrm{rad}(X,Y)$ by Proposition 17.2. If $f \in \mathrm{rad}^2(X,Y)$, then we claim that $f$ can be written as $f = hg$ where $g \in \mathrm{rad}(X,Z)$ and $h \in \mathrm{rad}(Z,Y)$ for some $Z$. In order to see this, observe that by definition, $f$ is a finite linear combination of $h_i g_i$ where $g_i \in \mathrm{rad}(X,Z_i)$ and $h_i \in \mathrm{rad}(Z_i,Y)$ for some $Z_i$. But if $f = h_1g_1 + h_2g_2$ then we may write $f$ as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\binom{g_1}{g_2}"'] & & Y \\ & Z_1 \oplus Z_2 \arrow[ur, "(h_1{,}h_2)"'] & \end{tikzcd}$$

and you can check that $\binom{g_1}{g_2}$ and $(h_1, h_2)$ belong to the respective radicals (note, though, that the $Z_i$ are not assumed to be indecomposable). Continuing inductively on the number of summands in $f = \sum h_i g_i$ yields $f = hg$ where $g \in \mathrm{rad}(X, Z)$ and $h \in \mathrm{rad}(Z, Y)$ for some $Z$, as required.

Now, since $f$ is irreducible, either $h$ is a split epimorphism or $g$ is a split monomorphism. If $g$ is a split monomorphism then there is some $u : Z \to X$ with $ug = 1_X$ and so $1_X - ug = 0$ which contradicts $g \in \mathrm{rad}(X, Z)$. So $h$ is a split epimorphism. But then there is a $v : Y \to Z$ with $hv = 1_Y$. But $hv \in \mathrm{rad}(Y, Y)$ by Lemma 17.3, so $1_Y \in \mathrm{rad}(Y, Y)$, a contradiction. Thus, $f \notin \mathrm{rad}^2(X, Y)$.

Conversely, suppose $f \in \mathrm{rad}(X, Y) \setminus \mathrm{rad}^2(X, Y)$. Since $X$ and $Y$ are indecomposable and $f$ is not an isomorphism, $f$ cannot be either split mono or split epi. Now suppose $f = hg$ where

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

and write $Z = \bigoplus Z_i$ as a direct sum of indecomposables $Z_i$. Let $j_i : Z_i \to Z$ be the $i^{th}$ inclusion and let $\pi_i : Z \to Z_i$ be the $i^{th}$ surjection. Then $f = \sum h_i g_i$ where $h_i = hj_i$ and $g_i = \pi_i g$. Some $h_i$ or $g_i$ must be an isomorphism or else $f \in \mathrm{rad}^2$. But if $h_i$ is an isomorphism then $h(j_i h_i^{-1}) = 1_Y$ and so $h$ is split epi, and if $g_i$ is an isomorphism then $g_i^{-1}\pi_i g = 1_X$ and so $g$ is split mono. Thus, $f$ is irreducible. $\square$

Thanks to Shisen for considerably simplifying the above proof!

**Definition 18.2.** *If $X$ and $Y$ are indecomposable modules, then we define*

$$\mathrm{Irr}(X, Y) := \mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$$

*and call it "the space of irreducible maps from $X$ to $Y$" (an incredibly bad name, because it isn't!)*

We see that the spaces whose dimensions give the number of arrows in the AR quiver are related to the number of maps which can't be factorised.

18.1. **Auslander-Reiten sequences.** Now we are ready to define AR sequences.

**Definition 18.3.** *An* Auslander-Reiten sequence *is a short exact sequence*

$$0 \longrightarrow X \xrightarrow{g} M \xrightarrow{f} Y \longrightarrow 0$$

*in which $X$ and $Y$ are indecomposable modules and $f$ and $g$ are irreducible maps.*

Because irreducible maps are scarce, so are Auslander-Reiten sequences. Note that if $Y$ is projective then there cannot be an AR sequence ending in $Y$. Similarly, if $X$ is injective then there cannot be an AR sequence ending in $X$.

A famous theorem of Auslander and Reiten states:

**Theorem 18.4** (Auslander, Reiten 1975)**.** *If $Y$ is a non-projective indecomposable module then there exists an indecomposable module $\tau Y$ and an AR sequence*

$$0 \to \tau Y \to M \to Y \to 0.$$

*Furthermore ,if*

$$0 \to Z \to N \to Y \to 0$$

*is an AR sequence, then $Z \cong \tau Y$ and $N \cong M$. (In fact, the two sequences are isomorphic.)*

The dual theorem is also true: if $X$ is a non-injective indecomposable then there is an indecomposable module $\tau^{-1}X$ and an AR sequence

$$0 \to X \to M \to \tau^{-1}X \to 0$$

etc.

The module $\tau Y$ appearing in the Auslander-Reiten Theorem is called the *Auslander-Reiten translate.* It can be defined via homological algebra, and we describe how to do this in the next lecture.

## 19. Lecture 19

Today's topic is the Auslander-Reiten translate. There are several ingredients that go into this. One is the functor $D : A - \mathrm{mod} \to \mathrm{mod} - A$ which was defined above as $DX = \mathrm{Hom}_k(X, k)$. We also want to use another functor $A - \mathrm{mod} \to \mathrm{mod} - A$ defined as $\mathrm{Hom}_A(-, A)$. We now list some properties of this functor.

19.1. **Homming into A.** If $M$ is a left $A$–module then $\mathrm{Hom}_A(M, A)$ is a right $A$–module. The right $A$–module structure is defined by: if $\phi \in \mathrm{Hom}_A(M, A)$ and $a \in A$ then $(\phi \cdot a)(m) = \phi(m)a$. Similarly, if $M$ is a right $A$–module then $\mathrm{Hom}_A(M, A)$ is a left $A$–module.

For any $A$–module $N$, there is a natural map

$$\eta_N : N \to \mathrm{Hom}_A(\mathrm{Hom}_A(N, A), A)$$

where the first $\mathrm{Hom}_A$ means homomorphisms of *right* $A$–modules. This natural map is defined by $n \in N \mapsto (\phi_n : \psi \mapsto \psi(n))$. In other words, it is given by "evaluation at $n$". It is easy to check that $\eta_A : A \to \mathrm{Hom}_A(\mathrm{Hom}_A(A, A), A)$ is an isomorphism of left $A$–modules. Since Hom preserves finite direct sums, $\eta_{A^{\oplus n}}$ is also an isomorphism for any $n$.

If $P$ is projective then there exists $Q$ with $P \oplus Q = A^{\oplus n}$. So

$$\eta_{P\oplus Q} : P \oplus Q \to \mathrm{Hom}_A(\mathrm{Hom}_A(P, A), A) \oplus \mathrm{Hom}_A(\mathrm{Hom}_A(Q, A), A)$$

is an isomorphism. But $\eta_{P\oplus Q}$ is given by

$$\begin{bmatrix} \eta_P & 0 \\ 0 & \eta_Q \end{bmatrix}$$

and therefore $\eta_P$ is also an isomorphism.

We conclude that $\mathrm{Hom}_A(-, A)$ induces a contravariant equivalence of categories

$$A - \mathrm{proj} \to \mathrm{proj} - A$$

where $\mathrm{proj} - A$, $A - \mathrm{proj}$ denote the categories of finite-dimensional right and left $A$–modules respectively.

19.2. **Minimal presentations.** Now we discuss *minimal presentations* of a module. If $M$ is a module, then let $P_0 \to M \to 0$ be a projective cover. Let $P_1 \to \ker(P_0 \to M)$ be a projective cover. Then the exact sequence

$$P_1 \to P_0 \to M \to 0$$

is called a *minimal presentation* of $M$. You can continue the process forever and get what is called a *minimal resolution*, but we are only interested in the $P_1$ and $P_0$ terms.

Given a left $A$-module $M$, take a minimal presentation

$$P_1 \to P_0 \to M \to 0$$

and let $\beta : P_1 \to P_0$ be the map in this sequence. Apply $\mathrm{Hom}_A(-, A)$ to obtain a map of right $A$–modules

$$\beta^* : \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A).$$

**Definition 19.1.** *The* transpose *of $M$ is defined to be* $\mathrm{Tr}(M) := \mathrm{cok}(\beta^*)$.

*The* Auslander-Reiten translate *of $M$ is the left $A$–module $\tau M := D\mathrm{Tr}(M)$.*

**Examples 19.2.** (1) Suppose $P$ is projective. Then a minimal presentation of $P$ is

$$0 \to 0 \to P \to P \to 0$$

and $\mathrm{Tr}(P) = \mathrm{cok}(\mathrm{Hom}_A(P, A) \to 0) = 0$.

(2) Suppose $A$ is hereditary. Then if $M$ is a left $A$–module, and $P_0 \to M$ is a projective cover, then $\ker(P_0 \to M)$ is projective. So there is an exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

which is a minimal presentation of $M$. Applying $\mathrm{Hom}_A(-, A)$ and writing down the long exact Ext sequence yields

$$0 \to \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A) \to \mathrm{Ext}^1_A(M, A) \to 0$$

because $P_0$ is projective and so $\mathrm{Ext}^1(P_0, -) = 0$. Thus, for a hereditary algebra $A$, we have $\mathrm{Tr} = \mathrm{Ext}^1_A(-, A)$ and so Tr is a functor in this case. In general, Tr need *not* be a functor.

The main reason for being interested in Tr is that it takes indecomposable modules to indecomposable modules. We have the following theorem.

**Theorem 19.3.** *Let $M$ be a left $A$–module.*

(1) $\mathrm{Tr}(M) = 0$ *if and only if* $M$ *is projective.*

(2) *If* $M$ *is indecomposable and not projective then* $\mathrm{Tr}(\mathrm{Tr}(M)) \cong M$.

(3) *If* $M$ *is indecomposable and not projective then* $\mathrm{Tr}(M)$ *is indecomposable.*

*Proof.* For the first part, we have already shown that $M$ projective $\implies$ $\mathrm{Tr}(M) = 0$. Conversely, suppose $\mathrm{Tr}(M) = 0$. Then if $P_1 \to P_0 \to M \to 0$ is a minimal presentation, then because $\mathrm{Hom}_A(-, A)$ is a left exact functor, we get an exact sequence

$$0 \to \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A) \to 0.$$

Since $\mathrm{Hom}(P_i, A)$ is projective for $i = 0, 1$, the sequence splits, and so there exists $\lambda : \mathrm{Hom}_A(P_1, A) \to \mathrm{Hom}_A(P_0, A)$ with $\beta^*\lambda = \mathrm{id}$, where $\beta : P_1 \to P_0$ is the map appearing in the minimal presentation of $M$. Taking the dual $\mathrm{Hom}_A(-, A)$ again, we obtain $\lambda^*\beta = \mathrm{id}$ which implies that $\beta$ is injective and the sequence

$$0 \longrightarrow P_1 \xrightarrow{\beta} P_0 \longrightarrow M \longrightarrow 0$$

splits. So $M$ is a summand of $P_0$, hence is projective.

For the second part, suppose $M$ is indecomposable and not projective. Then $\mathrm{Tr}(M) \neq 0$. Writing $P_i^*$ as shorthand for $\mathrm{Hom}_A(P_i, A)$, $\mathrm{Tr}(M)$ is defined as the cokernel of

$$P_0^* \xrightarrow{\beta^*} P_1^* \xrightarrow{\pi} \mathrm{Tr}(M) \longrightarrow 0 \tag{2}$$

where

$$P_1 \xrightarrow{\beta} P_0 \longrightarrow M \longrightarrow 0$$

is a minimal presentation of $M$. We wish to show that (2) is a minimal presentation of $\mathrm{Tr}(M)$. All the books say that this is obvious, but it isn't obvious to me. The best argument I could come up with is the following (based on [ASS06]). Let $Q_1$ be the projective cover of $\mathrm{Tr}(M)$. Then $P_1^* \twoheadrightarrow Q_1$ and so $P_1^* \cong Q_1 \oplus Q'$ for some $Q'$. Furthermore, if $\pi : P_1^* \to \mathrm{Tr}(M)$ denotes the projection, then $Q' \subset \ker(\pi)$ (this follows from the definition of a projective cover). We have $P_0^* \twoheadrightarrow \ker(\pi)$ and therefore if $Q_2$ is the projective cover of $\ker(\pi) = \ker(\pi|_{Q_1})$ then $P_0^* \cong Q_2 \oplus Q''$ for some $Q''$. This yields a commutative diagram

$$\begin{tikzcd}
Q_2 \arrow[r] \arrow[d, shift left] & Q_1 \arrow[r] \arrow[d, shift left] & \mathrm{Tr}(M) \arrow[r] \arrow[d, equal] & 0 \\
Q_2 \oplus Q'' \arrow[r, "\beta^*"] \arrow[u, shift left] & Q_1 \oplus Q' \arrow[r] \arrow[u, shift left] & \mathrm{Tr}(M) \arrow[r] & 0
\end{tikzcd}$$

where the top row is a minimal presentation of $\mathrm{Tr}(M)$ and the vertical maps are the projections and inclusions. The map $\beta^*$ can be written as a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where we regard elements of the direct sums as column vectors. Commutativity of the diagram implies that $c = d = 0$. Thus, $\beta = (\beta^*)^* = \left(\begin{smallmatrix} a^* & 0 \\ 0 & d^* \end{smallmatrix}\right) : Q_1^* \oplus (Q')^* \to Q_2^* \oplus (Q'')^*$ and $M = \mathrm{cok}(\beta) = \mathrm{cok}(a^*) \oplus \mathrm{cok}(d^*)$. But $M$ is indecomposable, so either $Q_2^*$ or $(Q'')^*$ surjects onto $M$. These are both projective submodules of $P_0$, which is the smallest projective module which surjects onto $M$.

Thus, either $Q_2 = 0$ or $Q'' = 0$. In either case, we obtain $Q' = 0$ as well, and so (2) is a minimal presentation of $\mathrm{Tr}(M)$, as required. Thus, by definition, $\mathrm{Tr}(\mathrm{Tr}(M)) \cong \mathrm{cok}((\beta^*)^*) = \mathrm{cok}(\beta)$.

For the last part, suppose $\mathrm{Tr}(M) = X \oplus Y$. It is an exercise to show that if $P_X \to X$ and $P_Y \to Y$ are projective covers, then $P_X \oplus P_Y \to X \oplus Y$ is also a projective cover. Thus, if

$$P_1 \xrightarrow{\beta} P_0 \longrightarrow M \longrightarrow 0$$

is a minimal presentation of $M$, then because we have shown that (2) is a minimal presentation of $\mathrm{Tr}(M)$, it follows that (2) has the form

$$Q_X \oplus Q_Y \to P_X \oplus P_Y \to X \oplus Y \to 0.$$

Thus, $\beta$ is given by a matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, where $x : P_X^* \to Q_X^*$ and $y : P_Y^* \to Q_Y^*$, and $M = \mathrm{cok}(\beta)$ is decomposable, a contradiction. □

**Example 19.4.** Let $A = kQ$ where $Q$ is the quiver $\bullet \longrightarrow \bullet$ . Label the vertices $0 \longrightarrow 1$ . There are three indecomposable $A$–modules; the simples $S_0$ and $S_1$, and the projective cover $P_0 = k \xrightarrow{1} k$ of $S_0$. Since $S_1$ and $P_0$ are projective, the only module with a nontrivial Auslander-Reiten translate is $S_0$. In order to calculate $\tau S_0$, we begin with the following minimal presentation of $S_0$

$$0 \to S_1 \to P_0 \to S_0 \to 0.$$

Now $S_1 = Ae_1$ and so $\mathrm{Hom}_A(S_1, A) = \mathrm{Hom}_A(Ae_1, A) \cong e_1A$. Similarly, $\mathrm{Hom}_A(S_0, A) = e_0A$. So $\mathrm{Tr}(S_0) = \mathrm{cok}(e_0A \to e_1A)$. This has dimension vector $\underline{\dim}(e_1A) - \underline{\dim}(e_0A) = (1,1) - (1,0) = (0,1)$. Taking the dual just means reversing the arrows, so we also have $\underline{\dim}(\tau S_0) = (0,1)$ and therefore $\tau S_0 = S_1$.

## 20. Lecture 20

In order to understand Auslander-Reiten sequences, we need to look at them from a different viewpoint. In this lecture, we will introduce almost-split sequences (which are short exact sequences which satisfy an a priori stronger property) and then show that they are the same as AR sequences.

### 20.1. **Almost-split and minimal morphisms.**

**Definition 20.1.** *$f : M \to N$ is called* left almost-split *if $f$ is not a split monomorphism and if*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "u"] & N \\ M' & \end{tikzcd}$$

*with $u$ not a split monomorphism, then there exists a $u' : N \to M'$ with $u'f = u$.*

*Dually, $g : M \to N$ is called* right almost-split *if $g$ is not split epi and if*

$$\begin{tikzcd} & N' \arrow[d, "v"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

*with $v$ not split epi, then there exists $v' : N' \to M$ with $gv' = v$.*

We also have the notion of minimality.

**Definition 20.2.** *$f : M \to N$ is called* left minimal *if for all $h : N \to N$ with $hf = f$, we have $h$ is an isomorphism.*

*Dually, $g : M \to N$ is called* right minimal *if for all $h : M \to M$ with $gh = g$, we have $h$ is an isomorphism.*

Finally, we say that $f : M \to N$ is *left minimal almost-split* if $f$ is left minimal and left almost-split. Similarly, we have the notion of *right minimal almost-split*. In the lectures, I often abbreviated these to *l.m.a.s* and *r.m.a.s*.

The notion of minimal morphism is rather intuitive, but the notion of almost-split morphism is rather mysterious. One way of looking at it is as a kind of non-split morphism which nevertheless behaves a bit like a split morphism. Indeed, if we dropped the requirement "$f$ is not split mono" from the definition of left almost-split, then any split monomorphism would be left almost-split. Similarly for right almost-split.

**Lemma 20.3.** *If $g : M \to N$ is right almost-split then $N$ is indecomposable.*

*If $f : M \to N$ is left almost-split then $M$ is indecomposable.*

*Proof.* Suppose $g : M \to N$ is right almost-split and $N = N_1 \oplus N_2$ with $N_i \neq 0$. Then the insertion maps $i_j : N_j \hookrightarrow N_1 \oplus N_2$ are not split epi, and so there exists $v_1 : N_1 \to N$ with $gv_1 = i_1$ and $v_2 : N_2 \to N$ with $gv_2 = i_2$. A simple computation then shows that $g(v_1 + v_2) = \mathrm{id}_N$, which contradicts that $g$ is not split epi.

The proof of the other part is the dual argument (reverse all arrows). $\square$

**Lemma 20.4.** *If $g : M \to N$ is left minimal almost-split or right minimal almost-split then $g$ is irreducible.*

*Proof.* If $g : L \to M$ is right-minimal almost-split then $M$ is indecomposable by the previous lemma, and $g$ is not split epi by definition. If $g$ is split mono then $M \cong L \oplus \mathrm{cok}(g)$ and so $g$ would be an isomorphism, hence split epi, a contradiction. So $g$ is neither split epi nor split mono.

Now suppose $g = g_1 g_2$ where

$$\begin{tikzcd} L \arrow[rr, "g"] \arrow[dr, "g_2"'] & & M \\ & N \arrow[ur, "g_1"'] & \end{tikzcd}$$

and $g_1$ is not split epi. Then there exists $v : N \to L$ with $gv = g_1$. Then $gvg_2 = g_1 g_2 = g$ and so by minimality, $vg_2 : L \to L$ is an automorphism. Thus, $g_2$ is split mono.

The argument for $g$ left-minimal almost-split is dual, again. $\square$

**Definition 20.5.** *A short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

*is called an* almost split sequence *if $f$ is left minimal almost-split and $g$ is right minimal almost-split.*

We shall see later that being left or right minimal almost-split is a strictly stronger condition than being irreducible (and that proving the existence of left or right minimal almost-split morphisms is quite hard). Nevertheless, we have the following remarkable theorem.

**Theorem 20.6.** *A short exact sequence is almost split if and only if it is an AR sequence.*

*Proof.* We have just shown that every almost split sequence is AR because minimal almost-split morphisms are irreducible, and the end terms are indecomposable by Lemma 20.3.

Suppose

$$0 \longrightarrow M'' \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$$

is an AR sequence. We wish to show that $g$ is right minimal almost-split. To this end, suppose $v : N \to M'$ is not split epi. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r, "f"] & M \arrow[r, "g"] & M' \arrow[r] & 0 \\
0 \arrow[r] & M'' \arrow[u, equal] & & N \arrow[u, "v"] \arrow[r] & 0.
\end{tikzcd}$$

It is a standard fact from homological algebra that the bottom row can be completed to a short exact sequence. Namely, let $U$ be the *pullback*

$$U := \{(m, n) \in M \oplus N | gm = vn\}.$$

Then there are maps $\alpha, \beta, \gamma$ such that the following diagram commutes, and the bottom row is exact.

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r, "f"] & M \arrow[r, "g"] & M' \arrow[r] & 0 \\
0 \arrow[r] & M'' \arrow[u, equal] \arrow[r, "\gamma"] & U \arrow[u, "\alpha"] \arrow[r, "\beta"] & N \arrow[u, "v"] \arrow[r] & 0.
\end{tikzcd}$$

If you haven't seen this before, you should work out explicitly what these maps are and check exactness. But it doesn't matter; all that matters is that such maps exist and that the bottom row is exact.

From the diagram, we have $f = \alpha\gamma$. Since $f$ is irreducible, either $\alpha$ is split epi or $\gamma$ is split mono.

If $\gamma$ is split mono then the bottom sequence splits, and so there exists $\lambda : N \to U$ with $\beta\lambda = \mathrm{id}_N$. So $g\alpha\lambda = v\beta\lambda = v$.

If $\alpha$ is split epi, then there exists $\lambda : M \to U$ with $\alpha\lambda = \mathrm{id}_M$. So $v\beta\lambda = g\alpha\lambda = g$. Therefore, we have $g = v\beta\lambda$ and $v$ is not split epi. Since $g$ is irreducible, $\beta\lambda$ must be split mono. So there exists $\xi : N \to M$ with $\xi\beta\lambda = \mathrm{id}_M$. Thus, the following short exact sequence splits

$$0 \longrightarrow M \xrightarrow{\beta\lambda} N \longrightarrow \mathrm{cok}(\beta\lambda) \longrightarrow 0$$

and $N \cong M \oplus \mathrm{cok}(\beta\lambda)$. Assume $N$ is indecomposable. Then $\beta\lambda$ is an isomorphism. Therefore, $g(\beta\lambda)^{-1} = v\beta\lambda(\beta\lambda)^{-1} = v$.

We have shown that if $N$ is indecomposable and $v : N \to M'$ is not split epi then there exists $v' : N \to M$ with $gv' = v$. Now let $N$ be arbitrary and $v : N \to M'$ not split epi. Then $N = \bigoplus N_i$ where the $N_i$ are indecomposable. If $v_i : N_i \to M'$ denotes the restriction of $v$ to $N_i$, then $v_i$ is not split epi and so there exists $v_i' : N_i \to M$ with $gv_i' = v_i$. Then $g\sum v_i' = \sum v_i = v$ as desired. So $g$ is right almost-split.

Now we show that $g$ is right minimal. Suppose $h : M \to M$ with $gh = g$. Then we have the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r, "f"] & M \arrow[r, "g"] & M' \arrow[r] & 0 \\
0 \arrow[r] & M'' \arrow[r, "f"] & M \arrow[r, "g"] \arrow[u, "h"] & M' \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

An easy diagram chase yields a map $\beta : M'' \to M''$ such that the following diagram commutes. (Don't believe me? Check it yourself!)

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r, "f"] & M \arrow[r, "g"] & M' \arrow[r] & 0 \\
0 \arrow[r] & M'' \arrow[r, "f"] \arrow[u, "\beta"] & M \arrow[r, "g"] \arrow[u, "h"] & M' \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Since $g$ is irreducible and $gh = g$ but $g$ is not split epi, we get that $h$ is split mono. Thus, $hf$ is a monomorphism and so if $x \in M''$ with $\beta(x) = 0$ then $f\beta(x) = hf(x) = 0 \implies x = 0$. Thus, $\beta$ is a monomorphism. By finite-dimensionality, $\beta$ is an isomorphism. Therefore, $h$ is also an isomorphism by the five lemma.

We have shown that $g$ is right minimal almost-split. The proof that $f$ is left minimal almost-split is, as usual, the dual argument. Thus, the given sequence is almost split. □

**Lemma 20.7.** *If $0 \to X \to M \to Y \to 0$ and $0 \to X' \to M' \to Y' \to 0$ are almost split sequences, then the following are equivalent.*

(1) *The two sequences are isomorphic (ie. there is a commutative diagram of the following form with the vertical maps isomorphisms).*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & M \arrow[r] & Y \arrow[r] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[u] & M' \arrow[r] \arrow[u] & Y' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

(2) $X \cong X'$.

(3) $Y \cong Y'$.

*Proof.* Suppose $Y$ is a module and $g : M \to Y$, $g' : M' \to Y$ are right minimal almost-split morphisms. Then there exists $u : M' \to M$ with $gu = g'$ and there exists $v : M \to M'$ with $g'v = g$. So $guv = g$ and $g'vu = g'$. Thus, $uv$ and $vu$ are isomorphisms, by minimality. It follows that $u$ and $v$ are isomorphisms. The five lemma now yields an isomorphism $\ker(g') \to \ker(g)$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & \ker(g) \arrow[r] & M \arrow[r, "g"] & Y \arrow[r] & 0 \\
0 \arrow[r] & \ker(g') \arrow[r] \arrow[u] & M' \arrow[r, "g'"] \arrow[u] & Y \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

This shows that any two almost-split sequences ending in the same module $Y$ are isomorphic, which is the statement (3) $\implies$ (1). The proof that (2) $\implies$ (1) is similar. $\square$

Lemma 20.7 is the uniqueness part of the Auslander-Reiten Theorem: an AR sequence is determined uniquely by either of its end terms. Notice that we had to use the notion of almost-split sequence instead of AR sequence in order to prove it.

The proof of Lemma 20.7 contains the following statement, which is a very useful observation which is worth singling out.

**Lemma 20.8.** *If $X$ is any module then there is, up to isomorphism, at most one right minimal almost-split morphism $g : M \to X$ (meaning that if $g' : N \to X$ is any other right minimal almost-split morphism then there is an isomorphism $N \to M$ making the following diagram commute).*

$$\begin{tikzcd}
M \arrow[r, "g"] & X \\
N \arrow[u] \arrow[ur, "g'"'] &
\end{tikzcd}$$

*Dually, there is at most one left minimal almost-split morphism $f : X \to M'$.*

20.2. **Identifying almost-split sequences.** Sometimes it is useful to have other characterisations of almost-split sequences, in order to reduce the amount of work needed to check that a given sequence is almost split. Here is one.

**Theorem 20.9.** *A short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

*is almost-split if and only if $g$ is right almost-split and $X$ is indecomposable if and only if (dually) $f$ is left almost-split and $Y$ is indecomposable.*

*Proof.* Suppose $g$ is right almost-split and $X$ is indecomposable. We need to show that $g$ is right minimal, and then show that $f$ is left minimal almost-split.

To show that $g$ is right-minimal, suppose that $h : M \to M$ with $gh = h$. Then, as in earlier proofs in this lecture, there exists $\beta : X \to X$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] & M \arrow[r, "g"] & Y \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[u, "\beta"] & M \arrow[r, "g"] \arrow[u, "h"] & Y \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

We want to show that $h$ is an isomorphism. If $\beta$ is an isomorphism, then $h$ is an isomorphism by the five lemma. Suppose $\beta$ is not an isomorphism. We want to reach a contradiction. Since $X$ is indecomposable, Fitting's Lemma implies that $\beta$ is nilpotent, so $\beta^m = 0$ for some $m$. Therefore, $h^m f = f\beta^m = 0$. Therefore, $h^m(\ker(g)) = 0$ and so there exists $u : Y \to M$ with $ug = h^m$ (the existence of such a $u$ is just the First Isomorphism Theorem). Since $gh = g$, we get $gug = gh^m = g$ and so $(gu - 1_Y)g = 0$. Since $g$ is surjective, this gives $gu = 1_Y$ and so $g$ is split epi, which contradicts that $g$ is right almost-split. Therefore, $h$ must be an isomorphism, and so $g$ is minimal.

The above argument was taken from [ASS06]. At this point, I ran out of time and so the rest of the proof will be given in the next lecture. □

## 21. Lecture 21

Last time I was in the middle of trying to prove a theorem, but we'll get back to that.

Recall that last time we proved that almost split sequences = AR sequences. We also proved that an almost-split sequence is uniquely determined by each of its end terms. This is the uniqueness part of the Auslander-Reiten Theorem. The existence part is the following.

**Theorem 21.1** (Auslander-Reiten)**.** *If $X$ is indecomposable and not projective then there exists an AR sequence*

$$0 \to \tau X \to M \to X \to 0$$

*and dually, if $Y$ is indecomposable and not injective then there exists an AR sequence*

$$0 \to Y \to N \to \tau^{-1} Y \to 0$$

*where $\tau^{-1}Y := \mathrm{Tr}(DY)$.*

*Proof.* We don't give the full proof, but just a sketch of the proof given in [ASS06], because some aspects of this proof are very important.

The proof relies on the fact that if $X, Y$ are modules then $\mathrm{Ext}^1_A(X,Y)$ may be identified with the space of equivalence classes of short exact sequences of the form

$$0 \to Y \to Z \to X \to 0.$$

Nobody in the class had seen this before, so I give a brief explanation. We define an equivalence relation on short exact sequences of the form $\delta_Z := 0 \to Y \to Z \to X \to 0$ by $\delta_Z \sim \delta_W$ if there exists an isomorphism $Z \to W$ making the following diagram commute.

$$\begin{tikzcd} & & Z \arrow[dd] \arrow[dr] & & \\ 0 \arrow[r] & Y \arrow[ur] \arrow[dr] & & X \arrow[r] & 0 \\ & & W \arrow[ur] & & \end{tikzcd}$$

There is an addition on the set of equivalence classes given by *Baer sum.* If $\delta_Z := 0 \to Y \to Z \to X \to 0$ and $\delta_W := 0 \to Y \to W \to X \to 0$ are short exact sequences, then we form their direct sum

$$0 \to Y \oplus Y \to Z \oplus W \to X \oplus X \to 0$$

and then take the pullback along the diagonal map $\Delta : Y \to Y \oplus Y$ followed by the pushout along the addition map $X \oplus X \to X$, to get a short exact sequence that begins with $Y$ and ends with $X$. This makes the set of equivalence classes of short exact sequences into an abelian group with identity element the class of the split sequence $0 \to Y \to Y \oplus X \to X \to 0$. This group happens to be isomorphic to $\mathrm{Ext}^1_A(X,Y)$ (this is in fact the reason for the name Ext). Therefore, $\mathrm{Ext}^1_A(X,Y)$ is a vector space. The scalar multiplication is given explicitly as follows. If $\lambda \in k \setminus \{0\}$ and $\delta_Z := \; 0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} X \longrightarrow 0$ is a short exact sequence, then $\lambda[\delta_Z]$ is the class of the sequence $0 \longrightarrow Y \xrightarrow{\lambda f} Z \xrightarrow{g} X \longrightarrow 0$. All these facts are proved in [Rot09], and I would recommend having a look at that book.

Anyway, back to the proof. A homological algebra argument can be used to show that for any two modules $M$ and $N$, there is an isomorphism of vector spaces

$$\mathrm{Ext}^1_A(M,N) \cong D\underline{\mathrm{Hom}}(\tau^{-1}N, M),$$

where $D$ denotes the linear dual as before, and $\underline{\mathrm{Hom}}(X,Y)$ is the quotient space of $\mathrm{Hom}(X,Y)$ by the subspace of all maps $X \to Y$ which factor through a projective module. The space $\underline{\mathrm{Hom}}(X,Y)$ can be zero, for example if one of the modules $X, Y$ happens to be projective.

If $X$ is non-projective and indecomposable, then the above formula gives an isomorphism

$$\mathrm{Ext}^1_A(X, \tau X) \cong D\underline{\mathrm{Hom}}(X,X),$$

because $\tau^{-1}\tau(X) \cong \mathrm{Tr}(\mathrm{Tr}(X)) \cong X$. Now, the space $\underline{\mathrm{Hom}}(X,X)$ is not 0, because the identity map $\mathrm{id}_X : X \to X$ does not factorise through a projective. Indeed, you can check that if $\mathrm{id}_X$ does factor through some projective $P$, then $X$ is a summand of $P$ and so $X$ is projective.

Thus, for $X$ indecomposable and nonprojective, $\mathrm{id}_X$ defines a nonzero element of $\mathrm{Ext}^1_A(X,\tau X)$, which yields an exact sequence

$$0 \to \tau X \to M \to X \to 0$$

for some $M$. To finish the proof, one can show that this is an AR sequence via further arguments using homological algebra. $\square$

**Corollary 21.2.** *If $X$ is a non-projective brick then any non-split sequence $0 \to \tau X \to M \to X \to 0$ is an AR sequence.*

*Proof.* If $X$ is a brick then by definition $\mathrm{End}(X) = k$. So $\dim D\underline{\mathrm{Hom}}(X,X) = 1 = \dim \mathrm{Ext}^1_A(X,\tau X)$. Any non-split short exact sequence $0 \to \tau X \to M \to X \to 0$ defines a non-zero element of $\mathrm{Ext}^1_A(X,\tau X)$, and hence is a scalar multiple of an AR sequence, and therefore is itself an AR sequence. $\square$

**Example 21.3.** This is a continuation of Example 19.4.

Let $Q = \bullet^0 \longrightarrow \bullet^1$ . We showed that $\tau S_0 = S_1$. Since $S_0$ is a brick and there is a nonsplit sequence $0 \to S_1 \to P_0 \to S_0 \to 0$, this sequence must be an AR sequence. It is the only AR sequence over $kQ$.

21.1. **Proof of Theorem 20.9.** Let $0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$ be a short exact sequence with $X$ indecomposable and $g$ right almost-split. We already showed that $g$ is right minimal. Now we have to show that $f$ is left minimal almost-split.

By the Auslander-Reiten Theorem, there exists an AR sequence $0 \longrightarrow X' \xrightarrow{f'} M' \xrightarrow{g'} Y \longrightarrow 0$ . By Lemma 20.8, there is an isomorphism $M' \to M$. We can then find a map $\beta$ between the kernels of $g'$ and $g$ so that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f'"] \arrow[d, "\beta"] & M' \arrow[r, "g'"] \arrow[d] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & M \arrow[r, "g"] & Y \arrow[r] & 0
\end{tikzcd}$$

The five lemma implies that $\beta$ is an isomorphism, and therefore $f$ is left minimal almost split since $f'$ is. This completes the proof of the theorem.

21.2. **The middle term of an AR sequence.** We are almost ready to start calculating examples of AR quivers. Before that, we need to understand the middle term of an AR sequence.

The following lemma explains the relationship between irreducible and minimal almost-split morphisms.

**Lemma 21.4.** *Let $Y$ be indecomposable and non-projective and let $f : X \to Y$. Then the following are equivalent.*

(1) $f$ *is irreducible.*

(2) *There exists* $f' : X' \to Y$ *such that* $f + f' : X \oplus X' \to Y$ *is right minimal almost-split.*

**Note:** The map $f + f'$ is defined by $(f + f')(x, x') = f(x) + f'(x')$. It is also often denoted $(f, f')$.

*Proof.* (1) $\implies$ (2). Suppose $f$ is irreducible. By the AR Theorem, there exists $g : M \to Y$ with $g$ right minimal almost-split. Since $f$ is not split epi, the right minimal almost-splitness of $g$ implies that there exists $t : X \to M$ with $f = gt$. Since $g$ is not split epi, $t$ must be split mono. Therefore, the sequence

$$0 \longrightarrow X \xrightarrow{t} M \longrightarrow \operatorname{cok}(t) \longrightarrow 0$$

splits. We take $X' = \operatorname{cok}(t)$. Then $M \cong X \oplus X'$ and $g$ may be written as $f + u : X \oplus X' \to Y$.

(2) $\implies$ (1). Suppose $g = f + f' : X \oplus X' \to Y$ is right minimal almost-split. We want to show that $f$ is irreducible. If $f$ is split epi then so is $g$, so $f$ cannot be split epi. If $f$ is split mono then $Y$ is decomposable, so $f$ cannot be split mono. Now suppose $f = ab$ with $a$ not split epi, where

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "b"'] & & Y \\ & Z \arrow[ur, "a"'] & \end{tikzcd}$$

Then since $a$ is not split epi, there exists $\lambda : Z \to X \oplus X'$ with $g\lambda = a$. Now you can check explicitly that the following diagram commutes.

$$\begin{tikzcd} X \oplus X' \arrow[r, "\left(\begin{smallmatrix} b & 0 \\ 0 & 1_{X'} \end{smallmatrix}\right)"] \arrow[dr, "f+f'"'] & Z \oplus X' \arrow[r, "\lambda+1_{X'}"] \arrow[d, "a+f'"] & X \oplus X' \arrow[dl, "f+f'"] \\ & Y & \end{tikzcd}$$

By minimality of $g = f + f'$, $(\lambda + 1_{X'}) \circ \left(\begin{smallmatrix} b & 0 \\ 0 & 1_{X'} \end{smallmatrix}\right)$ must be an isomorphism. If we write $\lambda = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$, we conclude that $ub : X \to X$ is an isomorphism, so $b$ must be split mono. Therefore, $f$ is irreducible. $\square$

The dual of the above lemma is also true. Together the lemma and its dual say that irreducible morphisms to or from indecomposables are precisely the components of minimal almost-split morphisms.

Now here is the important result, which we will prove next time.

**Theorem 21.5.** [ASS06, Corollary 4.4] *Suppose*

$$0 \longrightarrow X \xrightarrow{f} M \longrightarrow Y \longrightarrow 0$$

*is an AR sequence. Let* $M = \bigoplus M_i^{n_i}$ *where the* $M_i$ *are pairwise nonisomorphic indecomposable modules. Write* $f$ *as* $\begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$ *where* $f_i : X \to M_i^{n_i}$ *and for each* $i$ *write* $f_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{pmatrix}$ *where* $f_{ij} : X \to M_i$.

*Then $f_{ij} \in \operatorname{rad}(X, M_i)$ for all $i$ and the images of the $f_{ij}$ in $\operatorname{Irr}(X, M_i) := \operatorname{rad}(X, M_i)/\operatorname{rad}^2(X, M_i)$ form a basis for each $i$. Furthermore, if $M'$ is an indecomposable module and $\operatorname{Irr}(X, M') \neq 0$ then $M' \cong M_i$ for some $i$.*

The above theorem, together with its dual, say the following. If

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

is an AR sequence, then

$$M \cong \bigoplus_{\substack{M' \text{ indec} \\ \operatorname{Irr}(X,M')\neq 0}} (M')^{\dim\operatorname{Irr}(X,M')} \cong \bigoplus_{\substack{N' \text{ indec} \\ \operatorname{Irr}(N',Y)\neq 0}} (N')^{\dim\operatorname{Irr}(N',Y)}$$

But remember that the number $\dim\operatorname{Irr}(X, Y)$ is (by definition) just the number of arrows from $X$ to $Y$ in the AR quiver. Therefore, the information contained in the AR quiver is precisely the modules, but not the morphisms, in every AR sequence over our algebra. **Writing down the AR quiver is precisely equivalent to writing down a list of all the AR sequences.** It's just that the quiver is a more compact way of recording the information, and also can be viewed as a kind of picture of the module category itself.

## 22. Lecture 22

22.1. **Proof of Theorem 21.5.** Let

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

be an AR sequence. Since $f$ is left minimal almost-split, we get that $f_{ij}$ is irreducible for all $i$, $j$, and so $f_{ij} \in \operatorname{rad}(X, M_i) \setminus \operatorname{rad}^2(X, M_i)$.

If $M'$ is indecomposable and $\operatorname{Irr}(X, M') \neq 0$ then there is an irreducible map $a : X \to M'$. Thus, there is a module $M''$ and $a'' : X \to M''$ such that $\left(\begin{smallmatrix} a \\ a'' \end{smallmatrix}\right) : X \to M' \oplus M''$ is left minimal almost-split. Uniqueness of left minimal almost-split morphisms from $X$ implies that $M' \oplus M'' \cong M$. Thus, $M'$ is an indecomposable summand of $M$, so is isomorphic to one of the $M_i$.

All that remains is to show that for a fixed $M_i$, $\{\overline{f}_{i1}, \ldots, \overline{f}_{in_i}\}$ is a basis of $\operatorname{Irr}(X, M_i)$, where $\overline{f}_{ij}$ denotes the image of $f_{ij}$ in $\operatorname{Irr}(X, M_i) = \operatorname{rad}(X, M_i)/\operatorname{rad}^2(X, M_i)$. Let $u \in \operatorname{rad}(X, M_i)$. We want to show that $u$ is a linear combination of the $f_{ij}$ modulo $\operatorname{rad}^2(X, M_i)$. First, $u$ is not split mono because otherwise $u$ could not be in the radical, by definition of the radical. So we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & M \\ M_i & \end{tikzcd}$$

with $u$ not split mono, and therefore there exists $t : M \to M_i$ with $tf = u$. Since $M = \bigoplus_i M_i^{n_i}$, we may write $t$ as $(t_1, \ldots, t_k)$ with $t_j : M_j^{n_j} \to M_i$. We may also write $t_j$ as $(t_{j1}, \ldots, t_{jn_j})$ with $t_{jr} : M_j \to M_i$.

Then $u = tf$ implies $u = \sum_j \sum_{p=1}^{n_j} t_{jp} f_{jp}$. Now, if $j \neq i$ then $t_{jp}$ is not an isomorphism since $M_i \ncong M_j$ by definition. So $t_{jp} \in \mathrm{rad}(M_j, M_i)$. Therefore, $t_{jp} f_{pj} \in \mathrm{rad}^2(X, M_i)$ and so, modulo $\mathrm{rad}^2$, we obtain

$$u = \sum_{p=1}^{n_i} t_{ip} f_{ip}.$$

Now $t_{ip} \in \mathrm{End}(M_i) = k\mathrm{id} + \mathrm{rad}(M_i, M_i)$ since the base field $k$ is algebraically closed and the module $M_i$ is indecomposable. Thus, modulo $\mathrm{rad}^2$, $u = \sum_{p=1}^{n_i} t_{ip} f_{ip} = \sum_p \lambda_{ip} f_{ip}$ with $\lambda_{ip}$ scalars, as required.

We have shown that $\{\overline{f}_{ij}\}$ are a spanning set of $\mathrm{Irr}(X, M_i)$. Now we need to show that they are linearly independent. Suppose

$$\sum \lambda_{ij} \overline{f}_{ij} = 0.$$

Then $\sum \lambda_{ij} f_{ij} \in \mathrm{rad}^2(X, M_i)$. If some $\lambda_{ij} \neq 0$ then the map $\gamma : M_i^{n_i} \to M_i$ defined by $(x_i) \mapsto \sum \lambda_{ij} x_i$ is split epi (a splitting is $x \mapsto (0, \ldots, 0, \lambda_{ij}^{-1} x, 0 \ldots, 0)$). Now, $\sum_j \lambda_{ij} f_{ij} = \gamma f_i$ is irreducible. This is because, since $f_i$ is a component of a minimal left almost-split morphism, so is $\gamma f_i$. Thus, $\sum_j \lambda_{ij} f_{ij} \notin \mathrm{rad}^2(X, M_i)$, a contradiction. So we must have $\lambda_{ij} = 0$ for all $i$ and $j$. This completes the proof.

**Remark 22.1.** It is also worth stating the dual of Theorem 21.5:

if

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

is an AR sequence and $M = \bigoplus M_i^{n_i}$ with the $M_i$ pairwise nonisomorphic indecomposables, and $g = (g_1, \cdots, g_k)$ with $g_i : M_i^{n_i} \to Y$, and $g_i = (g_{i1}, \ldots, g_{in_i})$ where $g_{ij} : M_i \to Y$, then the images $\overline{g}_{ij}$ , $1 \leq j \leq n_i$, form a basis of $\mathrm{Irr}(M_i, Y)$ for each $i$, and if $M'$ is indecomposable with $\mathrm{Irr}(M', Y) \neq 0$ then $M'$ is isomorphic to one of the $M_i$.

22.2. **The Auslander-Reiten quiver.** To recap: the AR quiver is defined as the quiver with

- vertices = isomorphism classes $[X]$ of indecomposables.
- # arrows $[X] \to [Y] = \mathrm{dimIrr}(X, Y)$.
- $\tau : \{\text{vertices}\} \to \{\text{vertices}\}$ a partially-defined function given by $\tau[X] = [\tau X]$ for $X$ not projective.

**Definition 22.2.** *A (possibly inifinite) quiver $Q$ is called a* translation quiver *if*

(1) *$Q$ has no loops.*
(2) *$Q$ is* locally finite, *ie. each vertex $x$ of $Q$ has only finitely many arrows going into it and only finitely many arrows coming out of it.*
(3) *There is a bijection $\tau : Q_0 \setminus A \to Q_0 \setminus B$ where $Q_0$ denotes the set of vertices of $Q$ and $A, B \subset Q_0$, with the property that if $x$ is a vertex such that $\tau x$ exists, then for every $y$, the number of arrows from $y$ to $x$ equals the number of arrows from $\tau x$ to $y$.*

The third condition means that part of the quiver might look like this:

$$
\begin{tikzcd}
& \tau x \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & & x \arrow[ll, dashed] \\
\tau y \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] & & y \arrow[ll, dashed] \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] &
\end{tikzcd}
$$

where the dashed arrows represent $\tau$. In other words, $\tau$ maps the quiver exactly onto itself, except that some bits might fall off the edge. In particular, for every $x$ and $y$, the number of arrows from $x \to y$ equals the number of arrows $\tau x \to \tau y$, provided $\tau x$ and $\tau y$ are both defined.

The AR quiver of an algebra is a translation quiver. It has no loops, because if $f : X \to X$ is irreducible, then $f$ is either injective or surjective, so $f$ is an isomorphism, a contradiction. It is locally finite, because the only $[Y]$ for which there can be an arrow $[X] \to [Y]$ are the indecomposable summands of the middle term of the AR sequence starting at $X$, of which there are only finitely many. To show property (3), let $A$ be the set of projective modules and $B$ the set of injective modules. Then $\tau : Q_0 \setminus A \to Q_0 \setminus B$ has an inverse $\tau^{-1} := \mathrm{Tr}D$. The statement about the number of arrows is the following proposition.

**Proposition 22.3.** *If $X$ and $Y$ are indecomposable and $X$ is not projective and then there is an isomorphism of vector spaces*

$$\mathrm{Irr}(Y, X) \cong \mathrm{Irr}(\tau X, Y).$$

*Proof.* We apply Theorem 21.5 and its dual. By the AR Theorem, there is an AR sequence

$$0 \to \tau X \to M \to X \to 0.$$

For every $Y$, the number of times $Y$ occurs as a summand of $M$ equals both the dimension of $\mathrm{Irr}(Y, X)$ and the dimension of $\mathrm{Irr}(\tau X, Y)$, as required. $\square$

## 23. Lecture 23

Today: only half a lecture. Calculations.

Recap: AR quiver.

- Vertices $Q_0$: $\{[X]\}$, set of isomorphism classes of indecomposables.
- # arrows $[X] \to [Y]$ is $\dim(\mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y))$.
- $\tau : Q_0 \setminus \{\text{projectives}\} \to Q_0 \setminus \{\text{injectives}\}$, $\tau[X] = [\tau X]$.

The quiver is locally finite, has no loops and is a translation quiver.

**Example 23.1.** Calculate the AR quiver of $\mathbb{C}Q$ where $Q$ is the quiver

$$\bullet^0 \longrightarrow \bullet^1 \longrightarrow \bullet^2$$

Indecomposables correspond to roots, that is, vectors $(a, b, c)$ such that $q(a, b, c) := a^2 + b^2 + c^2 - ab - bc = 0$, with $(a, b, c)$ a nonzero vector of positive integers. Let us write $abc$ as shorthand for $(a, b, c)$. Then the six roots are $100, 010, 001, 110, 011, 111$. We need to list the indecomposables. There are three simple modules: $S_0 = \ k \longrightarrow 0 \longrightarrow 0\ $, $S_1 = \ 0 \longrightarrow k \longrightarrow 0\ $ and $S_2 = \ 0 \longrightarrow 0 \longrightarrow k\ $. The module $S_2$ is projective, and we have the projective covers $P_0 = (\ k \xrightarrow{1} k \xrightarrow{1} k\ ) \twoheadrightarrow S_0$ and $P_1 = (\ 0 \longrightarrow k \xrightarrow{1} k\ ) \twoheadrightarrow S_1$. Finally there is the injective envelope $I_1 = \ k \xrightarrow{1} k \longrightarrow 0\ $ of $S_1$. All of these modules are indecomposable, and we know by Kac' Theorem that there is a bijection between indecomposable modules and roots. So we have the complete list.

Since $P_0, P_1$ and $S_2$ are projective, there are only three AR sequences. Also, we know that each indecomposable $X$ is a brick (see the proof of Gabriel's Theorem above) and therefore any nonsplit sequence $0 \to \tau X \to M \to X \to 0$ is an AR sequence. So we need to calculate the translate of $S_0, S_1$ and $I_1$. Let us begin with $S_0$.

The short exact sequence $0 \to P_1 \to P_0 \to S_0 \to 0$ is a resolution of $S_0$. We have $P_1 = Ae_1$ and $P_0 = Ae_0$, so $\mathrm{Tr}(S_0)$ is the cokernel of $e_0A \to e_1A$. Instead of calculating this explicitly, we can calculate its dimension vector, which is $\underline{\dim}(e_1A) - \underline{\dim}(e_0A) = 110 - 100 = 010$. Thus, $\underline{\dim}(\tau S_0) = 010$ and $\tau S_0 = S_1$. A similar calculation gives $\tau S_1 = S_2$. Finally, we can calculate $\tau I_1$ by using the resolution $0 \to S_2 \to P_0 \to I_1 \to 0$, and we get $\underline{\dim}(\tau I_1) = 111 - 100 = 011$ so $\tau I_1 = P_1$.

To find the AR sequences, we now just need to write down three nonsplit sequences of the form $0 \to \tau X \to Y \to X \to 0$. The first two of these are easy:

$$0 \to S_1 \to I_1 \to S_0 \to 0$$
$$0 \to S_2 \to P_1 \to S_1 \to 0$$

The third one has the form

$$0 \to P_1 \to \text{something} \to I_1 \to 0.$$

The middle term has dimension vector $\underline{\dim}(I_1) = \underline{\dim}(P_1) = 121$. It contains $S_1$ as a summand because we have just shown that there is an irreducible map from $P_1$ to $S_1$. It cannot contain $S_0$ or $S_2$ as a summand because $\mathrm{Hom}(S_0, I_1) = \mathrm{Hom}(S_2, I_1) = 0$. Therefore, the middle term must be $P_0 \oplus S_1$ and the third sequence is

$$0 \to P_1 \to P_0 \oplus S_1 \to I_1 \to 0.$$

Notice that we don't even need to know the maps in this sequence; the Auslander-Reiten Theorem is powerful enough to say that there *must* be a nonsplit exact sequence of this form.

An alternative way of displaying the three sequences calculated above is the AR quiver.

$$
\begin{tikzcd}
 & & {[P_0]} \arrow[dr] & & \\
 & {[P_1]} \arrow[ur] \arrow[dr] & & {[I_1]} \arrow[ll, dashed] \arrow[dr] & \\
{[S_2]} \arrow[ur] & & {[S_1]} \arrow[ll, dashed] \arrow[ur] & & {[S_0]} \arrow[ll, dashed]
\end{tikzcd}
$$

**Example 23.2.** Now let $Q = \bullet^0 \longrightarrow \bullet^1 \longleftarrow \bullet^2$ . What changes?

The number of indecomposables is still six, and their dimension vectors are the same, because the roots depend only on the underlying graph. But they correspond to different modules. We have the three simples $S_0, S_1, S_2$ and the projective covers $P_0 = ( \ k \xrightarrow{1} k \longleftarrow 0 \ ) \twoheadrightarrow S_0$ and $P_2 = ( \ 0 \longrightarrow k \xleftarrow{1} k \ ) \twoheadrightarrow S_2$. There is also the injective envelope of $S_0$, $S_0 \hookrightarrow I_0 = ( \ k \xrightarrow{1} k \xleftarrow{1} k \ )$. Following the same sort of argument as in the previous example, the AR quiver is as follows.

$$
\begin{tikzcd}
 & {[P_0]} \arrow[dr] & & {[S_2]} \arrow[ll, dashed] \\
{[P_1]} \arrow[ur] \arrow[dr] & & {[I_1]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & \\
 & {[P_2]} \arrow[ur] & & {[S_0]} \arrow[ll, dashed]
\end{tikzcd}
$$

Suppose we had been given this quiver and were asked to write down the AR sequences. We can just read them off from the quiver. They are:

$$0 \to P_0 \to I_1 \to S_2 \to 0,$$

$$0 \to P_2 \to I_1 \to S_0 \to 0,$$

and

$$0 \to P_1 \to P_0 \oplus P_2 \to I_1 \to 0.$$

## 24. Lecture 24

24.1. **The Grothendieck group.** We now take a break from Auslander-Reiten Theory and introduce the Grothendieck group. This could have been mentioned much earlier, and is quite an important idea.

**Definition 24.1.** *Let $A$ be a finite-dimensional algebra. The* Grothendieck group $K_0(A)$ *is the free abelian group on the set of isomorphism classes $[M]$ of finite-dimensional $A$–modules, modulo the relations $[M] = [X] + [Y]$ whenever there is a short exact sequence $0 \to Y \to M \to X \to 0$.*

The group $K_0$ can be defined for much more general objects than finite-dimensional algebras. We are about to show that $K_0(A)$ is not very interesting as a group. However, we will also see that it is a useful computational tool.

**Definition 24.2.** *If $M$ is an $A$–module and $S_i$ is simple, the symbol $[M : S_i]$ denotes the number of copies of $S_i$ in a composition series of $M$.*

The number $[M : S_i]$ is well-defined, by the Jordan-Hölder Theorem.

**Theorem 24.3.** *$K_0(A)$ is a free abelian group on the set $\{[S_i] : S_i \text{ simple}\}$.*

*Proof.* Let $S_1, \dots, S_N$ be the simple $A$–modules. We define a $\mathbb{Z}$–linear map $K_0(A) \to \mathbb{Z}^N$ by $[M] \mapsto ([M : S_i])_{i=1}^N$. This is a well-defined map from the free abelian group on the set of isomorphism classes of $A$–modules to $\mathbb{Z}^N$. It induces a map on the group $K_0(A)$ because if $0 \to Y \to M \to X \to 0$ is a short exact sequence, then $[M : S_i] = [X : S_i] + [Y : S_i]$ for any $S_i$. We define a map $\mathbb{Z}^N \to K_0(A)$ by $(a_i)_{i=1}^N \mapsto \sum_i a_i[S_i]$. The two maps we have defined are mutually inverse, because if $M$ is an $A$–module then in $K_0(A)$ we have $[M] = \sum [M : S_i][S_i]$. So $K_0(A) \cong \mathbb{Z}^N$. $\square$

**Definition 24.4.** *The* Cartan matrix *of $A$ is the $N \times N$ matrix*

$$C = ([P_i : S_j])_{i,j=1}^N$$

*where $S_1, \dots, S_N$ are the simple $A$–modules and $P_i$ is the projective cover of $S_i$.*

**Theorem 24.5.** *If $A$ has finite global dimension then $C$ is invertible.*

*Proof.* The statement that $A$ has finite global dimension is equivalent to saying that every simple module $S_i$ has a finite projective resolution

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to S_i \to 0.$$

By splitting this long exact sequence up into short exact sequences, it is easy to show that in the Grothendieck group, we get

$$[S_i] = \sum_i (-1)^i [Q_i].$$

Now, each $Q_i$ may be written as a direct sum of the $P_j$, and so we obtain $[S_i] = \sum_j b_{ij}[P_j]$ for some $b_{ij} \in \mathbb{Z}$. Since $[P_j] = \sum [P_j : S_k][S_k]$ and the $[S_k]$ are a $\mathbb{Z}$–basis of $K_0(A)$, we get that $(b_{ij}) \cdot C = I$. Thus, $C$ is an invertible matrix. $\square$

**Example 24.6.** Let $A = k[x]/(x^2)$. Then $A$ has one simple module $S$ (to see this, either express $A$ as a quiver with relations, or write down a composition series for $A$ and observe that there are two composition factors which are isomorphic to each other). The projective cover of $S$ is $A$ itself. Thus, the Cartan matrix is the $1 \times 1$ matrix $C = (2)$. This is not invertible, so $\operatorname{gldim}(A) = \infty$.

24.2. **The Coxeter transformation.** Now suppose $A = kQ$ is the path algebra of a quiver with no oriented cycles and with vertex set $Q_0 = \{1, 2, \dots, n\}$. We have already seen how to calculate the simple and indecomposable projective modules. If $S_i$ denotes the simple module with $k$ at the $i^{th}$ vertex and 0 at the other vertices, and $P_i$ denotes the projective cover of $S_i$, then for all $i, j$ we have $[P_i : S_j] = c_{ij}$ where $c_{ij}$ denotes the number of paths from $i$ to $j$. Therefore,

$$\underline{\dim}(P_i) = (c_{i1}, \dots, c_{in})^T = C^T \underline{\dim}(S_i)$$

where $C = (c_{ij})$ is the Cartan matrix.

Also, we have the injective envelope $I_i$ of $S_i$, which is obtained by taking the projective cover of the corresponding simple for the opposite quiver and then dualising. Its dimension vector is

$$\underline{\dim}(I_i) = (c_{1i}, \dots, c_{ni})^T = C \underline{\dim}(S_i)$$

Thus, we have

$$\underline{\dim}(I_i) = -(-C(C^T)^{-1})\underline{\dim}(P_i).$$

**Definition 24.7.** *If $Q$ is a cycle-free quiver with vertex set $\{1, 2, \dots, n\}$ and $c_{ij}$ denotes the number of paths in $Q$ from $i$ to $j$ ($c_{ii} = 1$) and $C$ denotes the $n \times n$ matrix $(c_{ij})$ then*

$$\text{cox} := -C(C^T)^{-1} : \mathbb{Z}^n \to \mathbb{Z}^n$$

*is called the* Coxeter transformation.

For a general algebra $A$ of finite global dimension, we define the Coxeter transformation $\text{cox} : K_0(A) \to K_0(A)$ by $\text{cox}([P_i]) = -[I_i]$ where $P_i$ is the projective cover of the $i^{th}$ simple $S_i$, and $I_i$ is the injective envelope of $S_i$. This makes sense because both $\{[P_i]\}$ and $\{[I_i]\}$ form a $\mathbb{Z}$–basis for $K_0(A)$. This agrees with the above definition in the quiver case because it is easy to check that for a path algebra $A = kQ$, the map

$$\underline{\dim} : K_0(A) \to \mathbb{Z}^n$$

is an isomorphism. In fact, it is the isomorphism defined in Theorem 24.3.

The reason for being interested in the Coxeter transformation at this point is given by the following theorem.

**Theorem 24.8.** *If $M$ is a non-projective $kQ$–module then*

$$\underline{\dim}(\tau M) = \text{cox}(\underline{\dim}(M)).$$

*Proof.* Let $M$ be a $kQ$ module and let $0 \to Q_1 \to Q_0 \to M \to 0$ be a minimal presentation of $M$ ($Q_1 \to Q_0$ is injective because $A$ is hereditary). If $P_1, \dots, P_n$ are the indecomposable projectives then

$Q_1 = \bigoplus q_i P_i$ and $Q_0 = \bigoplus r_i P_i$ for some $q_i, r_i \geq 0$. Also, we have $\mathrm{Hom}_A(P_i, A) = \mathrm{Hom}_A(Ae_i, A) = e_i A$ and so $\underline{\dim}\mathrm{Hom}_A(P_i, A) = \underline{\dim}(e_i A) = \underline{\dim}(I_i)$. Therefore

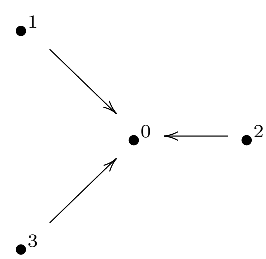
$$\begin{aligned}
\underline{\dim}(\tau M) &= \underline{\dim}\mathrm{Hom}_A(Q_0, A) - \underline{\dim}\mathrm{Hom}_A(Q_1, A) \\
&= \sum q_i \underline{\dim}(I_i) - \sum r_i \underline{\dim}(I_i) \\
&= \sum (q_i - r_i) \cdot (-\mathrm{cox}(\underline{\dim}(P_i))) \\
&= \mathrm{cox} \sum (r_i - q_i) \underline{\dim}(P_i) \\
&= \mathrm{cox}(\sum r_i \underline{\dim}(P_i) - \sum q_i \underline{\dim}(P_i)) \\
&= \mathrm{cox}(\underline{\dim}(Q_0) - \underline{\dim}(Q_1)) \\
&= \mathrm{cox}(\underline{\dim}(M)).
\end{aligned}$$

□

The above theorem is very useful for calculating the Auslander-Reiten quiver of the path algebra of a quiver of Dynkin (ADE) type. By Kac' Theorem, such a quiver has a unique indecomposable module of dimension $r$ for each root $r$ of the quiver. Since the AR translate of a non-projective indecomposable module $M$ is indecomposable, Theorem 24.8 enables us to calculate $\tau$ without having to do anything except matrix multiplications. Indeed, for such an $M$, $\tau M$ is the unique indecomposable module of dimension $\mathrm{cox}(\underline{\dim}(M))$.

## 25. Lecture 25

**Example 25.1.** Compute the AR quiver of $\mathbb{C}Q$ where $Q$ is the following quiver




of type $D_4$.

To do this, we first need to list the indecomposable modules for $\mathbb{C}Q$. The ones we can write down straight away are the simple, indecomposable projective and indecomposable injective modules. The simple modules are $S_0, S_1, S_2, S_3$. Only $S_0$ is projective. The others have nontrivial projective covers $P_i$. Dually, $S_1, S_2$ and $S_3$ are injective, but $S_0$ has a nontrivial injective envelope $I_0$. We have thus found eight indecomposable modules.

Because $Q$ is a Dynkin quiver, Kac' Theorem tells us that an indecomposable module is uniquely determined by its dimension vector, so we will usually identify modules with their dimension vectors. In this notation, we write the modules as follows.

$$S_0 = {}^{0}_{0}\,{\scriptstyle 1\ 0} \qquad\qquad I_0 = {}^{1}_{1}\,{\scriptstyle 1\ 1}$$

$$S_1 = {}^{1}_{0}\,{\scriptstyle 0\ 0} \qquad\qquad P_1 = {}^{1}_{0}\,{\scriptstyle 1\ 0}$$

$$S_2 = {}^{0}_{0}\,{\scriptstyle 0\ 1} \qquad\qquad P_2 = {}^{0}_{0}\,{\scriptstyle 1\ 1}$$

$$S_3 = {}^{0}_{1}\,{\scriptstyle 0\ 0} \qquad\qquad P_3 = {}^{0}_{1}\,{\scriptstyle 1\ 0}$$

This is not the complete set of indecomposables, as we shall see. We can find some more by taking the AR translates of the modules in the above list. The Cartan matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and the Coxeter transformation is

$$\mathrm{cox} = -C(C^T)^{-1} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

By applying the Coxeter transformation to the dimension vectors of each of the modules found above, we obtain

$$\tau S_1 = {}^{0}_{1}\,{\scriptstyle 1\ 1} \quad \tau S_2 = {}^{1}_{1}\,{\scriptstyle 1\ 0} \quad \tau S_3 = {}^{1}_{0}\,{\scriptstyle 1\ 1} \quad \tau I_0 = {}^{1}_{1}\,{\scriptstyle 2\ 1}$$

We have now found 12 distinct indecomposable modules. Looking up the root system of $D_4$, it has 12 positive roots, and thus we have the complete set of indecomposables. The translates of the modules we haven't calculated yet are:

$$\tau\left({}^{0}_{1}\,{\scriptstyle 1\ 1}\right) = P_1 \quad \tau\left({}^{1}_{1}\,{\scriptstyle 1\ 0}\right) = P_2 \quad \tau\left({}^{1}_{0}\,{\scriptstyle 1\ 1}\right) = P_3 \quad \tau\left({}^{1}_{1}\,{\scriptstyle 2\ 1}\right) = S_0.$$

Now we know $\tau$, we just need to write down the AR sequences.

For $S_1$, we have a sequence

$$0 \to {}^{0}_{1}\,{\scriptstyle 1\ 1} \to (?) \to S_1 \to 0.$$

The middle term has dimension vector ${}^{1}_{1}\,{\scriptstyle 1\ 1}$ and it cannot have a simple summand because there are no irreducible maps between two simple modules. From the list of indecomposables, we see that the only possibility for this module is $I_0$. A similar argument for $S_2$ and $S_3$ gives the three AR sequences:

$$0 \to \begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix} \to I_0 \to S_1 \to 0$$
$$0 \to \begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix} \to I_0 \to S_2 \to 0$$
$$0 \to \begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix} \to I_0 \to S_3 \to 0.$$

Next, we have an AR sequence

$$0 \to P_1 \to (?) \to \begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix} \to 0.$$

The middle term has dimension vector $\begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}$. What can it be? We know from the AR sequences already calculated that there is an irreducible map $\begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix} \to I_0 = \tau^{-1}\left(\begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}\right)$. Neither of these modules is projective, and therefore, since $\dim \mathrm{Irr}(X,Y) = \dim \mathrm{Irr}(\tau X, \tau Y)$ when $X$ and $Y$ are nonprojective, there is also an irreducible map $P_1 = \tau\left(\begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix}\right) \to \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}$. Therefore, $\begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}$ is a summand of the middle term (?), and hence is the whole of the middle term. So we get another three AR sequences.

$$0 \to P_1 \to \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix} \to \begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix} \to 0$$
$$0 \to P_2 \to \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix} \to \begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix} \to 0$$
$$0 \to P_3 \to \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix} \to \begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix} \to 0.$$

There are just two more AR sequences to calculate. Their middle terms are completely determined by the sequences we have already written down, and we see that they are forced to have the form

$$0 \to S_0 \to P_1 \oplus P_2 \oplus P_3 \to \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix} \to 0$$

and

$$0 \to \begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix} \to \begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix} \to I_0 \to 0.$$

Our eight sequences may be recorded in the AR quiver.

$$\begin{tikzcd}
 & {[P_1]} \arrow[dr] & & \left[\begin{smallmatrix}0&&\\&1&1\\1&&\end{smallmatrix}\right] \arrow[ll, dashed] \arrow[dr] & & {[S_1]} \arrow[ll, dashed] \\
{[S_0]} \arrow[ur] \arrow[dr] \arrow[ddr] & & \left[\begin{smallmatrix}1&&\\&2&1\\1&&\end{smallmatrix}\right] \arrow[ll, dashed] \arrow[ur] \arrow[dr] \arrow[ddr] & & {[I_0]} \arrow[ll, dashed] \arrow[ur] \arrow[dr] \arrow[ddr] & \\
 & {[P_2]} \arrow[ur] & & \left[\begin{smallmatrix}1&&\\&1&0\\1&&\end{smallmatrix}\right] \arrow[ll, dashed] \arrow[ur] & & {[S_2]} \arrow[ll, dashed] \\
 & {[P_3]} \arrow[uur] & & \left[\begin{smallmatrix}1&&\\&1&1\\0&&\end{smallmatrix}\right] \arrow[ll, dashed] \arrow[uur] & & {[S_3]} \arrow[ll, dashed]
\end{tikzcd}$$

25.1. **The first Brauer-Thrall Theorem.** We don't have time to explore the applications of Auslander-Reiten theory in detail, but we will explain the following theorem, which Auslander-Reiten theory was invented to solve. The theorem is called the *first Brauer-Thrall theorem* because it was a famous conjecture of Brauer and Thrall.

**Theorem 25.2** (Auslander, Reiten)**.** *If $A$ is a finite-dimensional algebra and there exists $N \in \mathbb{N}$ such that $\dim(M) < N$ for all indecomposable $A$–modules $M$ then $A$ is of finite type.*

The theorem shows, for example, that the path algebra of the quiver $Q = \bullet \rightrightarrows \bullet$ is not of finite type, because it has infinitely many two-dimensional indecomposable modules. However, we already knew this from Gabriel's Theorem anyway. The theorem does tell us, however, that $kQ$ has indecomposable modules of arbitrarily large dimension.

The proof of Theorem 25.2 is taken directly from [ASS06, IV.5]. It uses two lemmas.

**Lemma 25.3** (Harada, Sai)**.** *Let $b \in \mathbb{N}$ and let $M_i$, $1 \leq i \leq 2^b$, be indecomposable modules over a finite-dimensional algebra $A$. Suppose the length $\ell(M_i) \leq b$ for all $i$. Suppose there are maps $f_i : M_i \to M_{i+1}$ which are nonisomorphisms.*

*Then the composition $f_{2^b-1}f_{2^b-2} \cdots f_2 f_1 = 0$.*

The Harada-Sai lemma is proved with about a page of linear algebra. We won't give the proof, but it can be found in [ASS06, IV, 5.2]. The lemma is sensible because the $f_i$ in general map $M_i$ to something smaller. The $f_i$ can't all be injective because there is a bound on the length of the $M_i$. So you can have a chain of at most $b$ injective $f_i$, then the next $f_i$ must collapse some of $M_i$ down to something smaller, then you could have another chain of injective ones, and so on. But eventually you are forced to collapse down to zero.

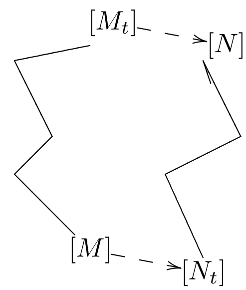
The second lemma is less intuitive.

**Lemma 25.4.** [ASS06, IV, 5.1] *Let $M, N$ be indecomposable modules with $\mathrm{Hom}(M, N) \neq 0$. Let $t \in \mathbb{N}$ and assume there is no path in the AR quiver from $M$ to $N$ of length $< t$.*

*Then there exist $M_0 \dots, M_t$ indecomposables with $M_0 = M$ and there exist irreducible maps $f_i : M_{i-1} \to M_i$ and a map $g : M_t \to N$ such that the composition $gf_t \cdots f_1 \neq 0$.*

*Similarly, there exist indecomposables $N_0 = N, N_1, \dots, N_t$ and irreducible maps $g_i : N_i \to N_{i-1}$ and $f : M \to N_t$ such that $g_1 \cdots g_t f \neq 0$.*

The lemma says that we can find a path of length $t$ in the AR quiver from $M$ to a module somewhat close to $N$, and a path of length $t$ from a module close to $M$ to $N$.




*Proof.* We just prove the second statement, the proof of the first being dual and given in [ASS06]. We prove the statement by induction on $t$. Suppose the statement is true for $t-1$ and that $\mathrm{Hom}(M,N) \neq 0$. Then we have a chain of indecomposable modules $N_i$ and irreducible morphisms $g_i : N_i \to N_{i-1}$

$$N_{t-1} \to N_{t-2} \to \cdots N_1 \to N_0 = N$$

and an $f : M \to N_{t-1}$ such that $g_1 \cdots g_{t-1} f \neq 0$. If $f$ were split epi, then by indecomposability of $M$, $f$ would have to be an isomorphism. But then we would have a chain of $t-1$ irreducible maps from $M$ to $N$, so there would be a path in the AR quiver of length $t-1$ from $M$ to $N$, a contradiction. Thus, $f$ is *not* split epi.

Now there are two cases: either $N_{t-1}$ is projective, or it is not.

Suppose first that $N_{t-1}$ is not projective. By the Auslander-Reiten Theorem, there exists a right minimal almost-split morphism $\theta : L \to N_{t-1}$. Since $f$ is not split epi, there is an $f' : M \to L$ with $\theta f' = f$. Now write $L = \bigoplus_{i=1}^{p} L_i$ with the $L_i$ indecomposable. Then $\theta = \theta_1 + \cdots + \theta_p$ with $\theta_i : L_i \to N_{t-1}$ and $f' = (f_1', \ldots, f_p')$ with $f_i' : M \to L_i$. So $\theta f' = \sum_{i=1}^{p} \theta_i f_i'$. Thus, since $g_1 \cdots g_{t-1} f \neq 0$, we get $\sum g_1 \cdots g_{t-1} \theta_i f_i' \neq 0$. Therefore, there is some $i$ with $g_1 \cdots g_{t-1} \theta_i f_i' \neq 0$. Now, $\theta_i$ is irreducible because $\theta$ was right minimal almost-split, and so we can take $g_t = \theta_i$ and replace $f$ by $f_i'$, which completes the inductive step.

Now suppose that $N_{t-1}$ is projective. Then $f$ is not epic, or it would be a split epimorphism. Since $N_{t-1}$ is an indecomposable projective, it has a unique maximal submodule $\mathrm{rad}(N_{t-1})$, and we must have $\mathrm{Im}(f) \subset \mathrm{rad}(N_{t-1})$. We have already shown that the inclusion $\mathrm{rad}(P) \hookrightarrow P$ is irreducible if $P$ is an indecomposable projective module (this is a good exercise if you don't remember why - it is just a direct application of the definition of an irreducible map). Write $\mathrm{rad}(N_{t-1}) = \bigoplus L_i$ with each $L_i$ indecomposable. Then each inclusion $L_i \hookrightarrow \mathrm{rad}(N_{t-1})$ is also irreducible, and therefore we may apply the same argument as in the first case to obtain the desired result. ☐

## 26. Lecture 26

Now we will complete the proof of the Brauer-Thrall theorem. It follows from the following stronger statement.

**Theorem 26.1.** *Let $(Q,\rho)$ be a quiver with relations and let $A = kQ/\langle\rho\rangle$. Suppose the quiver $Q$ is* connected*. Suppose the AR quiver of $Q$ has a connected component $C$ such that the length $\ell(M) \leq b$ for all $M$ in $C$. Then $C$ is finite and $C$ is the whole of the AR quiver.*

*Proof.* First, we prove the following statement:

**Claim:** *If $M, N$ are indecomposable $A$–modules with* $\mathrm{Hom}(M,N) \neq 0$ *and either $M \in C$ or $N \in C$, we claim that there is a path in the AR quiver from $M$ to $N$ of length $\leq 2^b - 1$.*

In order to see this, assume $N \in C$ and suppose there is no path from $M$ to $N$ of length $< t := 2^b - 1$. Now apply Lemma 25.4. This says that there exists a chain of indecomposable modules $N_i$ and irreducible maps $g_i : N_i \to N_{i-1}$, and a map $f : M \to N_t$, with $N_0 := N$ and $g_1 \cdots g_t f \neq 0$. Since $N \in C$ and there is a path in the AR quiver from $N_i$ to $N$, each $N_i \in C$ as well. But then $\ell(N_i) \leq b$ for all $i$, and the Harada-Sai lemma immediately implies that $g_1 \dots g_t = 0$ because none of the $g_i$ is an isomorphism (they are irreducible maps). This contradicts $g_1 \cdots g_t f \neq 0$ and therefore there must be a path from $M$ to $N$ of length $2^b - 1$.

Similarly, if $M \in C$ then we can use the other half of Lemma 25.4 to reach the same conclusion. This proves the claim.

Now we show that $C$ is the whole of the AR quiver. Suppose $N \in C$. Then there is an indecomposable projective module $P$ with $\mathrm{Hom}(P,N) \neq 0$ (take a summand of a projective cover of $N$). So $P \in C$. Now since $A = kQ/\langle\rho\rangle$, $P = Ae_i$ for some vertex $i$ of $Q$. If $j$ is a vertex such that there is an arrow from $j$ to $i$, then $\mathrm{Hom}(Ae_i, Ae_j) \cong e_i A e_j \neq 0$ and therefore $Ae_j \in C$. Similarly, if there is an arrow from $i$ to $j$ then $Ae_j \in C$. Since $Q$ is connected, we conclude that $Ae_j \in C$ for every vertex $j$ of $Q$. Therefore, $C$ contains every indecomposable projective. Now if $M$ is any indecomposable module, then there is an indecomposable projective $P'$ with $\mathrm{Hom}(P', M) \neq 0$ and since $P' \in C$, we get $M \in C$, as required. Thus, $C$ is the whole AR quiver.

It remains to show that the AR quiver is finite. But the above argument shows that

$$C = \bigcup_{P \text{ indec projective}} \{M : \mathrm{dist}(M,P) \leq 2^b - 1\}$$

where dist denotes the shortest path in the underlying graph of the AR quiver. Since there are only finitely many indecomposable projective modules, this is a finite set. $\square$

26.1. **Proof of the Brauer-Thrall Theorem.** Let $A$ be an algebra with $\dim(M) \leq N$ for all indecomposable $A$–modules $M$. Then $\ell(M) \leq N$ for all indecomposable $A$–modules $M$. There exists a quiver with relations $(Q,\rho)$ such that $A$ is Morita equivalent to $kQ/\langle\rho\rangle$. Since length is clearly preserved by Morita equivalence, it follows that $\ell(M) \leq N$ for all indecomposable $kQ/\langle\rho\rangle$–modules $M$. Now assume that $A$ has only one block. Then $Q$ is connected, because otherwise $kQ/\langle\rho\rangle$ would have more than one block. The previous theorem now implies that $kQ/\langle\rho\rangle$ has finite type (take $C$ to be the whole AR quiver. Then the theorem shows that $C$ is finite), and therefore so does $A$.

If $A$ has more than one block, then the above argument shows that each block of $A$ has finite type. Therefore, so does $A$. This completes the proof.

26.2. **Structure of AR quivers.** Let us finish the course with some general discussion of what AR quivers look like, since so far we have only seen a few examples. First we give a theorem (again taken directly from [ASS06]) which shows that the AR quiver of an algebra of finite type cannot have parallel arrows.

**Theorem 26.2.** *Let $A$ be an algebra of finite type. Then the AR quiver of $A$ has no multiple arrows.*

*Proof.* Suppose $M$, $N$ are indecomposable $A$–modules with $\dim\mathrm{Irr}(M,N) \geq 2$. Then there exists an irreducible map $f : M \to N$. Because $f$ is irreducible, $f$ is either injective or surjective, and not both. Suppose $f$ is injective. The other case is dual, see [ASS06, Proposition 4.9].

If $f$ is injective, then $\dim(M) < \dim(N)$ and $f : M \hookrightarrow N$. Therefore, $M$ cannot be injective, or else $N$ would be isomorphic to $M \oplus \mathrm{cok}(f)$ and since $N$ is indecomposable, $f$ would be forced to be an isomorphism, a contradiction. So by the AR Theorem, there exists an AR sequence

$$0 \to M \to X \to \tau^{-1}M \to 0$$

and $X \cong N^{\oplus 2} \oplus U$ for some $U$, by the description of the middle term that we had.

Therefore, $\dim(\tau^{-1}M) + \dim(M) = 2\dim(N) + \dim(U)$ and so $\dim(\tau^{-1}M) \geq 2\dim(N) - \dim(M) > \dim(N) > \dim(M)$.

Now, from $X = N^{\oplus 2} \oplus U$, we get $\dim\mathrm{Irr}(N, \tau^{-1}M) \geq 2$, so there is an irreducible map $f : N \to \tau^{-1}M$. This map must be injective since $\dim(\tau^{-1}M) > \dim(N)$. So $N$ is not an injective module. Therefore, there is an AR sequence

$$0 \to N \to Y \to \tau^{-1}N \to 0$$

and $\dim(\tau^{-1}N) > \dim(\tau^{-1}M)$ by the same argument as above.

Continuing inductively we obtain, for every $i \geq 0$,

$$\dim(\tau^{-i-1}N) > \dim(\tau^{-i-1}M) > \dim(\tau^{-i}N) > \dim(\tau^{-i}M).$$

Therefore, $\{\tau^{-i}N\}$, $i \geq 0$ is a collection of infinitely many nonisomorphic indecomposables, which contradicts that $A$ has finite type. $\square$

So the AR quiver of an algebra of finite type cannot have multiple arrows. We have also seen that no AR quiver can have loops. What about cycles? It *is* possible to have cycles. For example, if $A = k[x]/(x^2)$, then $A$ has just two indecomposable modules, namely $A$ and $S := A/xA$ (we haven't proved this) and the only AR sequence is of the form

$$0 \to S \to A \to S \to 0.$$

Thus, the AR quiver is

$$\begin{tikzcd} {[A]} \arrow[r, bend left] & {[S]} \arrow[l, bend left] \arrow[loop right, dashed] \end{tikzcd}$$

which has a cycle.

For hereditary algebras $kQ$ of finite type, this can't happen. There are never cycles and the arrows tend to go from left to right. On the left, we have all the projectives, and on the right all the injectives (look back at the $D_4$ example computed above). It is a good exercise to show that an algebra is hereditary if and only if for every indecomposable projective $P$, if $[M] \to [P]$ is an arrow in the AR quiver, then $M$ is also projective. The dual statement for injectives is also true, which explains why all the projectives are at one end of the AR quiver and all the injectives are at the other.

When we started AR theory, I did mention that one motivation for the AR quiver was as a kind of picture of the module category. For algebras of finite type, the following theorem shows that it is a good picture.

**Theorem 26.3.** [ASS06, Corollary 5.6] *If $A$ is an algebra of finite type and $f : M \to N$ is a nonisomorphism between indecomposable $A$–modules, then $f$ is a sum of compositions of irreducible morphisms.*

Thus, there is a nontrivial morphism $M \to N$ if and only if there is a path from $M$ to $N$ in the AR quiver. Of course, the AR quiver doesn't tell us the whole module category because it doesn't contain information about how morphisms compose.

## 27. Lecture 27

What about algebras of infinite type? These fall into two classes, according to the following theorem.

**Theorem 27.1** (Drozd)**.** *An algebra $A$ of infinite type is either tame or wild.*

Tame *means that for every $N \in \mathbb{N}$, there exists a finite collection $M_1, \dots, M_{p(N)}$ of $A - k[x]$–bimodules such that every indecomposable module of dimension $< N$ is isomorphic to $M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $\lambda \in k$.*

Wild *means that there exists an $A$–$k\langle x, y\rangle$–bimodule $M$ such that the functor $F : k\langle x, y\rangle-\mathrm{mod} \to A-\mathrm{mod}$ defined by $F(N) = M \otimes_{k\langle x,y\rangle} N$ takes nonisomorphic objects to nonisomorphic objects.*

Roughly speaking, tame means that although there are infinitely many indecomposables, they fall into finitely many nice families, each one depending on a parameter. Wild means "at least as bad as $k\langle x, y\rangle$" which is bad.

The above theorem is known as *Drozd's tame-wild theorem.* It is a very famous result, and people put a lot of effort into determining whether a given algenra is tame or wild.

For hereditary algebras, this was solved by Nazarova.

**Theorem 27.2** (Nazarova)**.** *Let $Q$ be a quiver without oriented cycles. Then $kQ$ is tame if and only if $Q$ is a Dynkin or extended Dynkin quiver.*

Nazarova's Theorem is a generalisation of Gabriel's Theorem which extends things from the finite type case to the tame case.

We have already seen that we can calculate the AR quiver of a Dynkin quiver via a process like the one we used in our $D_4$ example. So what about an extended Dynkin quiver? Here, too, the AR quiver can be calculated explicitly. Of course, it is an infinite quiver because the algebra no longer has finite type. We do not give the proofs, but they can be found in the notes [CB]. Instead, let us describe what happens.

Consider one of the simplest cases, the $\widetilde{A}_2$ case. This is the quiver $\bullet \rightrightarrows \bullet$ , also known as the *Kronecker quiver*. The quadratic form of this quiver is $q(\alpha_0, \alpha_1) = (\alpha_0 - \alpha_1)^2$ and the roots are the vectors with $q(\alpha_0, \alpha_1) \leq 1$. Therefore, the dimension vectors of the indecomposables are $(a, a)$, $(a, a-1)$ and $(a-1, a)$ for $a \geq 1$. We can get some idea of what the components of the AR quiver are by using the Coxeter transformation. The Cartan matrix is

$$C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and the Coxeter transformation is

$$\mathrm{cox} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$

We see that $\mathrm{cox}\left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right)$, and in fact $\tau X = X$ for every indecomposable $X$ of dimension $(a, a)$. We already know that there is an indecomposable

$$k \underset{\mu}{\overset{\lambda}{\rightrightarrows}} k$$

for each $\lambda, \mu \in k$, and it is easy to see that these are isomorphic if and only if $(\lambda, \mu) = (r\lambda, r\mu)$ for some $r \in k$. It can be shown that each of these indecomposables is contained in a unique component of the AR quiver, and so there are $\mathbb{P}^1$ many of these components. They are called *tubes*.

We also note that $\mathrm{cox}\left(\begin{smallmatrix} a+1 \\ a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+3 \\ a+2 \end{smallmatrix}\right)$ and $\mathrm{cox}\left(\begin{smallmatrix} a \\ a+1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a-2 \\ a-1 \end{smallmatrix}\right)$. In fact, all the modules of dimension vector $(a, a+1)$ belong to a single component of the AR quiver. It looks like this.

$$\begin{tikzcd}
 & 12 \arrow[dr, shift left] \arrow[dr, shift right] & & 34 \arrow[ll, dashed] \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots \arrow[ll, dashed] \\
01 \arrow[ur, shift left] \arrow[ur, shift right] & & 23 \arrow[ll, dashed] \arrow[ur, shift left] \arrow[ur, shift right] & & 45 \arrow[ll, dashed] & & \cdots \arrow[ll, dashed]
\end{tikzcd}$$

It carries on to the right forever. Here, I have used the notation $ab$ as shorthand for a representation of dimension vector $(a, b)$. It turns out that if $a \neq b$ then there is a unique dimension vector $(a, b)$. Indeed, this follows from Kac' Theorem.

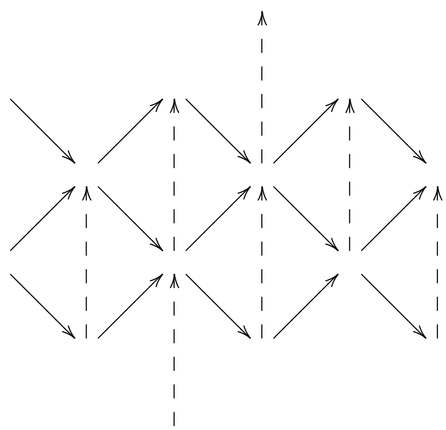
In this component, 01 is the projective simple and 12 is the projective cover of the other simple. Thus, this component of the AR quiver consists of all those indecomposable modules $M$ such that $\tau^i M$ is projective for some $i$. Such modules are called *preprojective* and this is called the *preprojective component*.

Dually, there is a *preinjective component* with the modules 10 and 21 on the left and all arrows going the other way.

The modules which are neither preinjective nor preprojective are called *regular*. Every such module $X$ has $\tau^i X = X$ for some $i$. The smallest such $i$ is called the *period*. For the Kronecker quiver, every regular module has period 1.

The above example is worked out in detail in [Hue, Section 5.3].

The picture looks the same for any $kQ$ when $Q$ is a Euclidean quiver. There is a preprojective component, a preinjective component, and a collection of tubes indexed by $\mathbb{P}^1$. The periods of all the modules $X$ in a given tube are the same, and this common value is called the period of the tube. The periods of different tubes can be different. A tube of period $> 1$ looks like this:



The picture should be thought of as filling the whole of the plane, and the dotted arrows represent the AR translate. Each point is identified with the point $i$ steps above it. Thus, the graph lies on a cylinder, hence the name "tube".

**Exercises 27.3.** Some exercises on the Grothendieck group and Auslander-Reiten theory.

(1) Show that $\mathrm{gldim}(\bigwedge \mathbb{C}^2) = \infty$.

(2) Let $Q$ be a quiver without cycles and $M$ a $\mathbb{C}Q$–module. Show that $M$ is projective if and only if all the entries of the vector $\mathrm{cox}(\underline{\dim}(M))$ are non-positive.

(3) Let $A = k[x,y]/(x^2, xy, y^2)$, a commutative algebra of dimension 3. Show that the following short exact sequence of $A$–modules is an AR sequence.

$$0 \longrightarrow A \xrightarrow{f} A/(x) \oplus A/(y) \xrightarrow{g} A/(x,y) \longrightarrow 0$$

where $f(a) = (a,a)$ and $g(p,q) = p - q$.

(4) Compute the AR quiver of $\mathbb{C}Q$ where $Q$ is the quiver

$$\begin{tikzcd}
\bullet^1 \arrow[dr] & & \\
& \bullet^0 \arrow[r] & \bullet^2 \\
\bullet^3 \arrow[ur] & &
\end{tikzcd}$$

(5) Let $A$ be a finite-dimensional algebra. Show that the following are equivalent.

(a) $A$ is hereditary.

(b) If $M$ is an indecomposable $A$–module and there is a projective indecomposable $P$ and an arrow $[M] \to [P]$ in the AR quiver of $A$, then $M$ is projective.

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